Lectures 4 and 5: Orbits

Reference Book:

A.E. Roy: Orbital Motion, 3rd edition 1988
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Orbital Motion

• Is the motion of a body around another one.
• Examples:
  – A solar system planet around the Sun
  – A star around a companion star (binary system)
  – A Galaxy around another (like LMC around the Milky Way)
  – A satellite around a planet
  – An artificial satellite around the Earth
• Basic experimental data come from observation of the orbits of the planets around the Sun.
• Kepler’s laws are very good approximations of precise astronomical measurements.

Kepler’s (empirical) laws

1. The orbit of each planet is an ellipse with the Sun at one focus
2. For any planet the rate of description of area by the radius vector joining the planet to Sun is constant
3. The cubes of the semimajor axes of the planetary orbits are proportional to the squares of the planets’ periods of revolution.

   ❑ At the time of their formulation (circa 1600) these laws described perfectly the observational data. Even today, with extremely precise data, these laws are a close first approximation to the truth. They also hold for various systems of satellites orbiting their primary.
   ❑ They fail only for close satellites of a nonspherical planet and for the outermost retrograde satellites.

Kepler’s laws explained

• Newton was the first one to explain these laws as a result of the laws of Dynamics and Gravitation.
• Kepler’s laws are a description of a special case of the solution of the gravitational problem of n bodies, where
  – All bodies can be treated as point-masses
  – All the masses but one are so small that they do not attract each other appreciably, but they are attracted solely by the large mass.
• These two conditions are verified quite well for the solar system planets orbiting around the Sun.
• We are now going to study this problem. The first thing to do is to define a suitable way to define an orbit in space. 6 quantities are needed.

6 “Elements” of the orbit in space

- xy = Reference Plane
  - Ecliptic for Planets;
  - Equator for Earth’s satellites
- Ω = Origin of longitude
- y = Vernal equinox

first element of the orbit: Ω

- N,N = Line of nodes
- N = Ascending node
- N = Longitude of the ascending Node
**second element of the orbit: \( i \)**

- \( i = \) inclination of the orbit

**third element of the orbit: \( \omega \)**

- \( \omega = \Omega + \omega = \) longitude of perihelion

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**4th, 5th, 6th elements: \( a, e \) and \( \tau \)**

- \( a = \) semimajor axis: \( A_1A_2 = 2a \)
- \( e = \) ellipticity: \( CS = ae = e \sqrt{1 - e^2} \)
- \( \tau = \) time of perihelion (perigee) passage
- \( R = \) \( SP = \) vector radius
- \( f = \) true anomaly, often used in addition

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**The two-body problem**

- Given at any time the positions and velocities of two known point-masses moving under their mutual gravitational force, compute their position and velocities at any other time.
- First solved by Newton.
- Needed for the solution:

\[
\frac{d}{dt} (m\vec{v}) = \vec{F}
\]

\[
\vec{F}_1 = G \frac{m_1 m_2 \vec{r}}{r^3} \quad ; \quad \vec{F}_2 = -G \frac{m_1 m_2 \vec{r}}{r^3}
\]

**The two-body problem**

- Motion equations:

\[
m_1 \frac{d^2 \vec{r}_1}{dt^2} = G \frac{m_1 m_2 \vec{r}}{r^3} \quad ; \quad m_2 \frac{d^2 \vec{r}_2}{dt^2} = -G \frac{m_1 m_2 \vec{r}}{r^3}
\]

**Motion of the centre of mass**

- \( m_1 \frac{d^2 \vec{r}_1}{dt^2} + m_2 \frac{d^2 \vec{r}_2}{dt^2} = 0 \)

- \( \vec{R}_C = \) def \( \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \)

- The centre of mass moves with constant velocity

\[
(m_1 + m_2) \vec{R}_C = \vec{a} \quad + \quad \vec{b}
\]
The two-body problem

- Motion equations: \( \frac{d^2\mathbf{r}}{dt^2} + G \frac{m_1}{r^3} \mathbf{r} = -G \frac{m_2}{r^3} \mathbf{r} \)
- Relative motion: subtract eqns. above:
  \( \mathbf{r} \times \left( \frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3} \mathbf{r} \right) = 0 \Rightarrow \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = 0 \)
- Take the vector product of \( r \) with equation above:
  \( \mathbf{r} \times \left( \frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3} \mathbf{r} \right) = 0 \Rightarrow \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = 0 \)
- Integrating by parts
  \( \mathbf{r} = \int \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} dt + \int \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} dt = \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} \)
- Where \( \mathbf{h} \) is a constant vector. The motion (orbit) lies in a plane perpendicular to \( \mathbf{h} \). (Angular momentum integral).

The two-body problem

- Relative motion equation: \( \frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3} \mathbf{r} = 0 \)
- Can be rewritten in polar coordinates:
  \( \mathbf{r} = r(t) \mathbf{e}_r + h(t) \mathbf{e}_\theta \)
- This is the mathematical form of Kepler’s second law: the motion along the orbit happens with constant area velocity.
- Integrating:
  \( \int r^2 dr dt + \int \mu r^2 dt = C \)
  \( \int r^2 dr dt + \int \mu r^2 dt = C \)
- This is related to energy conservation.
The two-body problem

- Substituting into\[ \frac{2}{u^2} \left( \frac{du}{dt} \right)^2 - \frac{1}{u} \frac{d}{dt} \left( \frac{1}{u} \right)^2 + \frac{\mu u}{b^2} = 0 \]

- We get: \[ \frac{d^2u}{dt^2} + \frac{\mu}{u} = 0 \]

- The solution is:
  \[ u = \frac{\mu}{b} + A \cos(\theta - \omega) \]

- With \( A \) and \( \omega \) constants. This is a conic section:

- So that

  - \( e = 0 \) circle
  - \( 0 < e < 1 \) ellipse
  - \( e = 1 \) parabola
  - \( e > 1 \) hyperbola

- Let’s study in detail the elliptic orbit.
- Relevant definitions / relations:

  - \( 2a = \) major axis
  - \( 2b = \) minor axis
  - \( CS \) / \( CA = e \)
  - \( b = \mu \sqrt{1 - e^2} \)
  - \( SP + S'P = 2a \)
  - \( x^2/a^2 + y^2/b^2 = 1 \)
  - \( f = \theta - \omega \) true anomaly
  - \( Q = a(1 - e) \)
  - \( Q = r(\pi / 2) = p \)
  - \( \theta = \omega \) perihelion
  - \( \theta = \pi + \omega \) aphelion

- The area of the ellipse is \( A = \pi ab \).
- The second law of Kepler can be written
  \[ \frac{dA}{dt} = \frac{h}{2} \]
  \[ h = \int_{0}^{T} \sqrt{\mu(1 - e^2)} \frac{r}{T} \]

- Moreover, the solution of the two-body problem gives
  \[ a(1 - e') = p = \frac{h^2}{\mu} \]
  \[ h = \sqrt{\mu a(1 - e')^3} \]

- So combining the two equations we have:
  \[ \frac{2ma}{h} = \sqrt{\mu a(1 - e')^3} \]

  - The period depends ONLY on the semimajor axis and on the sum of the two masses

### LEO satellite period

**Low-Earth Orbit**

\[
T = 2\pi \sqrt{\frac{a^3}{\mu}} = 2\pi \sqrt{\frac{(R_e + h)^3}{GM_e + m}}
\]

\[ T(\text{min}) = 1.659 \times 10^{-4} \text{[a}(\text{km})]^3 \]

### Satellite period for higher orbits

\[
T = 2\pi \sqrt{\frac{a^3}{\mu}} = 2\pi \sqrt{\frac{(R_e + h)^3}{GM_e + m}}
\]

\[ T(\text{min}) = 1.659 \times 10^{-4} \text{[a}(\text{km})]^3 \]
Elliptic Orbit

- As an immediate consequence we have that for two masses $m_1$ and $m_2$ orbiting around the same centre mass $M_s$

$$\frac{M_s + m_1}{M_s + m_2} = \left(\frac{a_1}{a_2}\right)^2 \frac{I_2}{I_1}$$

- This is the correct form of the third law of Kepler for the Solar System. Here

$$M_s \gg m_t \rightarrow \left(\frac{a_1}{a_2}\right)^2 \frac{I_2}{I_1} \equiv 1$$

Tycho Brahe (1546-1601)

- Tycho designed and built new instruments, calibrated them, and instituted nightly observations.
- Changed observational practice profoundly: earlier astronomers observed the positions of planets and the Moon at certain important points of their orbits (e.g., opposition, quadrature, station), Tycho observed these bodies throughout their orbits.
- As a result, a number of orbital anomalies never before noticed were made explicit by Tycho. Without these complete series of observations of unprecedented accuracy, Kepler could not have discovered that planets move in elliptical orbits.

Johannes Kepler (1571-1630)

- In 1609 he published Astronomia Nova, delineating his discoveries, which are now called Kepler's first two laws of planetary motion. In 1619 he published Harmonices Mundi, in which he describes his "third law."
- Kepler published the seven-volume Epitome Astronomiae in 1621. This was his most influential work and discussed all of heliocentric astronomy in a systematic way. He was a sustainer of the copernican system.

Velocity in an Elliptic Orbit

- Let $V$ be the velocity of the orbiting mass in the position rf.
- Since $\vec{r} = \vec{r} + \vec{r}'\vec{\theta}$
- The modulus of the velocity is thus $V^2 = \dot{r}^2 + \dot{r}^2 \dot{f}^2$
- It is possible to express this as a function of $r$ only. We use

$$r = \frac{p}{1 + \cos f} \text{ → } \dot{r} = \frac{p}{1 + \cos f} \dot{f} = \frac{h}{r^2} \sin f$$

$$r^2 \dot{f} = h \frac{\dot{r}}{r} = \frac{h}{r^2} \sin f$$

(1)

(2)

- Inserting (1) and (2) inside, $V^2 = \dot{r}^2 + \dot{r}^2 \dot{f}^2$ we get:

$$V^2 = \left(\frac{h}{p}\right)^2 \left[1 + 2\cos f \sin f\right] = \left(\frac{h}{p}\right)^2 \left[\cos f + 2\cos f \sin f\right] = \left(\frac{h}{p}\right)^2 \left[\cos f + 2\cos f \sin f + 1\right] = \left(\frac{h}{p}\right)^2 \left[2 + 2\cos f \left(1 - e^2\right)\right]$$

- This formula shows that the velocity depends only on the radius $r$ and allows the computation of the velocity in any point of the orbit:

- At pericenter $r = a(1 - e) \rightarrow V^2 = \left(\frac{h}{p}\right)^2 \left[2 + 2\cos f \left(1 - e^2\right)\right]$

- At apocenter $r = a(1 + e) \rightarrow V^2 = \left(\frac{h}{p}\right)^2 \left[2 + 2\cos f \left(1 - e^2\right)\right]$

- If a satellite is injected at distance $r$ from the center body and with initial velocity $V$, the semimajor axis of the orbit $a$ will not depend on the direction of the initial velocity.

Nicholas Copernicus (1473-1543)

In De Revolutionibus Orbium Coelestium ("On the Revolutions of the Celestial Orbs"), which was published in Nuremberg in 1543, the year of his death, stated that the Sun was the center of the universe and that the Earth had a triple motion around this center.

His theory gave a simple and elegant explanation of the retrograde motions of the planets (the annual motion of the Earth necessarily projected onto the motions of the planets in geocentric astronomy) and settled the order of the planets (which had been a convention in Ptolemy's work) definitively.
• If a satellite is injected at distance $r$ from the center body and with initial velocity $V$, the semimajor axis of the orbit $a$ will not depend on the direction of the initial velocity.
• Moreover, for all the orbits with different velocity directions and same $V$ and $r$, the period $T$ is also the same, because $T$ depends on $a$ only. So if 3 satellites are injected from $P$ along the 3 orbits, they will be back in $P$ at the same time.

**Velocity in an Circular Orbit**

- If the body has to move on a circular orbit, $r=constant=a$, so that
- $V^2 = \frac{\mu}{r}$
- But $V^2 \text{ circ} = \frac{\mu}{2a}$
- Moreover, for such an orbit $V^2 = \frac{\mu}{R_e}$

**Elliptic Orbits of Artificial Satellites**

- Artificial satellites are carried in orbit by multi-stage rockets. At the separation point, at an altitude $h$, the last stage of the rocket impresses to the satellite a velocity $V$.  
- In the figure we see what happens if the velocity at separation is orthogonal to the radius vector. If the velocity is too small, the satellite will fall to the ground on an elliptic section (1). Otherwise the separation point will be the apogee (2).
- The minimum velocity at separation needed to obtain a full orbit can be computed imposing that at apogee
- $V_{\text{apogee}}^2 = \frac{\mu}{R_e+h}$
- And at perigee
- $V_{\text{perigee}}^2 = \frac{\mu}{R_e}$
- Moreover, for such an orbit $2a = h + 2R_e$
- So the velocity at separation has to be $V_s^2 = \frac{\mu}{a} (1-e) = \frac{\mu}{R_e} (R_e+h)/2 R_e + h$

**Kepler’s equation**

- We first derive the relationship between the true anomaly $f$ and eccentric anomaly $E$:
  - $\frac{r}{r_0} = \cos E$
  - $a (1-e) \cos E = a (1-e) \cos f$
  - So $\cos E = \cos f$
  - But $r \cos f = a (\cos E - e)$
  - A property of ellipses and eccentric circles is that $PR / QR = b / a$
  - So $r \sin f = b$
  - $a \sin E = a \sqrt{1-e^2} \sin E$
  - $r \sin f = b \sin E = a \sqrt{1-e^2} \sin E$
Kepler’s equation

- Combining (1) and (2)
  \[ r^2(\cos^2 f + \sin^2 f) = r^2 = (a \cos E - ae^2) + a^2(1 - e^2) \sin^2 E = a^2(1 - e \cos E) ] \rightarrow r = a(1 - e \cos E) \\
- Moreover
  \[ \tan \frac{\n}{2} = \frac{a(1 + e)(1 - \cos E)}{a(1 - e)(1 + \cos E)} \]
  which is the relation between eccentric anomaly \( E \) and true anomaly \( f \).

Kepler’s equation

- Now we derive Kepler’s equation, which relates the mean anomaly \( M \) and the eccentric anomaly \( f \).
- By Kepler’s second law:
  \[ \frac{area \ SPA}{ab} = \frac{t - \tau}{T} \]
  Or
  \[ area \ SPA = \frac{M}{2} \]
- Now
  \[ area \ SPA = area \ SPR + area \ ARP \]
  But
  \[ area \ ARP = \frac{a}{b} \]
  so
  \[ area \ SPA = area \ SPR + \frac{a}{b} \]
  area \ ARP

Kepler’s equation

- Combining
  \[ area \ SPA = \frac{1}{2} \]
  \[ 2\pi f - \tau = M = E - esinE \]
- and
  \[ area \ SPA = \frac{M}{2} \]
- We finally get Kepler’s equation:

Kepler’s equation

- Now we have everything in place to solve the most common elliptic orbit problems, which are:
  - Get the position and velocity of the orbiting body, given the elements and the time
  - Obtain the elements of the orbit, given the position and velocity and the time
- The needed formulae are summarized here:

\[
\begin{align*}
  r &= \frac{a(1 - e^2)}{1 + e \cos f} \\
  h &= \mu a(1 - e^2) \\
  r &= a(1 - e \cos E) \\
  V &= \mu \left(2 - \frac{1}{a} \right) \\
  \tan \frac{f}{2} &= \frac{\sqrt{(1 + e) \tan E}}{\sqrt{(1 - e)}} \\
  T &= 2\pi / n \\
  M &= E - esinE \\
  n &= \mu^{\frac{1}{3}} \sqrt{\frac{2}{\mu}} \\
  M &= n(t - \tau) \\
  \sin \theta &= \frac{\sqrt{(1 - e^2)}}{(2a - r)}
\end{align*}
\]
Parabolic Orbit

- In this type of two-body motion the orbit is open. The second body approaches the first one from infinity and, after the nearest approach, recedes back to infinity.
- The equation of the orbit is obtained by putting $e=1$ in the conic equation:

$$r = \frac{a(1-e^2)}{1+e \cos f}$$  \hspace{1cm} (1)

$$\dot{h}^2 = \mu a(1-e^2)$$  \hspace{1cm} (6)

- In the Axy system the equation of the parabola is:

$$\tan \frac{f}{2} = \tan \frac{E}{2} \sqrt{\frac{1-e}{1+e}}$$  \hspace{1cm} (2)

$$V^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right)$$  \hspace{1cm} (7)

- The equation of the orbit is obtained by putting $e>1$ in the conic equation:

$$\tan \frac{f}{2} = \tan \frac{E}{2} \sqrt{\frac{1-e}{1+e}}$$  \hspace{1cm} (3)

$$T = \frac{2\pi}{a}$$  \hspace{1cm} (8)

- The equation of the orbit is obtained by putting $e=1$ in the conic equation:

$$M = E - e \sin E$$  \hspace{1cm} (4)

$$n = \mu^{1/2} e^{-3/2}$$  \hspace{1cm} (9)

- The equation of the orbit is obtained by putting $e=0$ in the conic equation:

$$M = n(t - \tau)$$  \hspace{1cm} (5)

$$\sin \varphi = \frac{\sqrt{(1-e^2)}}{\sqrt{(2a-r)}}$$  \hspace{1cm} (10)

Circular and parabolic velocity

- Consider a satellite orbiting in a circular orbit with velocity

$$V^2 = \frac{\mu}{r}$$

- Imagine the satellite is given an impulse so that it reaches the parabolic velocity in the same point:

$$V_{par} = 2\mu \frac{e}{r}$$

- The satellite will now follow a parabolic orbit that will take it to infinity. It will reach infinity with 0 velocity, so the parabolic velocity is the same as the escape velocity.

- If the impulse is larger, the satellite will go to infinity on a hyperbolic orbit.

Hyperbolic Orbit

- This orbit is also open. The second body approaches the first one from infinity and, after the nearest approach, recedes back to infinity.
- The equation of the orbit is obtained by putting $e=1$ in the conic equation:

$$r = \frac{a(1-e^2)}{1+e \cos f}$$  \hspace{1cm} (1)

$$\dot{h}^2 = \mu a(1-e^2)$$  \hspace{1cm} (6)

- In the Axy system the equation of the parabola is:

$$\tan \frac{f}{2} = \tan \frac{E}{2} \sqrt{\frac{1-e}{1+e}}$$  \hspace{1cm} (2)

$$V^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right)$$  \hspace{1cm} (7)

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$$\sin \varphi = \frac{\sqrt{(1-e^2)}}{\sqrt{(2a-r)}}$$  \hspace{1cm} (10)

Absolute and relative motions

- If G is centre of mass of the two masses $m_1$ and $m_2$, its position vector $R$ is

$$MR = m_1 \overrightarrow{R}_1 + m_2 \overrightarrow{R}_2 \quad ; \quad M = m_1 + m_2$$

- If we put the origin $O$ in G we see that

$$m_1 \overrightarrow{R}_1 = m_2 \overrightarrow{R}_2 = r = \overrightarrow{R}_1 + \overrightarrow{R}_2$$

- Where $r$ is the position of $m_2$ with respect to $m_1$. From the two equations above we get immediately

$$\overrightarrow{R}_1 = \frac{m_1}{M} r \quad ; \quad \overrightarrow{R}_2 = \frac{m_2}{M} r$$

- For the relative motion the second law of Kepler applies: $r \ddot{f} = h$, and since the two masses and the center of mass are always on a straight line, the same law must apply for the barycentric orbits as well: $\overrightarrow{R}_1 \ddot{f} = h_1 \quad ; \quad \overrightarrow{R}_2 \ddot{f} = h_2$
Absolute and relative motions

- From \( R_1 = \frac{m_1}{M} r \) ; \( R_2 = \frac{m_2}{M} r \) 
- we get 
  \[ h = \left( \frac{m_1}{M} \right)^2 \frac{r}{h} = \left( \frac{m_2}{M} \right)^2 \frac{h}{r} \] 
- The barycentric orbits are thus geometrically similar to each other and to their relative orbit.
- This means that they have same eccentricities and same periods, and that, for example 
  \[ a_1 = \frac{m_1}{M} a ; \quad a_2 = \frac{m_2}{M} a \]

Orbits and Energy

- We are now in a position to relate the shape of the orbits to the total energy of the system.
- If \( V_1 \) and \( V_2 \) are the velocities with respect to the center of mass (taken to be at rest), the total energy of the system is 
  \[ E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + G \frac{m_1 m_2}{r} \] 
- So the energy required in A to place the satellite in the final orbit (radius \( r \)) is 
  \[ \Delta E = \frac{1}{2} m_1 v_A^2 + \frac{1}{2} m_2 v_B^2 + G \frac{m_1 m_2}{r} - \frac{1}{2} m_1 v_1^2 - \frac{1}{2} m_2 v_2^2 \]
- Hence for a closed orbit the total energy (kinetic plus potential) must be negative, for escape just to take place the total energy is zero; for an energy \( > 0 \) an escape along a hyperbola takes place.

Transfer between Orbits

If motors are not used, the satellite will follow its conic orbit, for example a circle around the Earth.

- Firing a motor will cause changes in the orbit, affecting in general all 6 elements of the orbit.
- Here we assume that firing is short enough that only the velocity is changed, while the position is not: so we assume that the impulse is applied instantly to the satellite.
- We start assuming that we want to transfer between coplanar circular orbits.
- To do this we apply an impulse in A, producing a velocity variation \( \Delta v_A \), and inserting the satellite in the final orbit (radius \( a_B \)).
- So the energy required in A to place the vehicle in the correct elliptical transfer orbit is 
  \[ \Delta C = C_f - C_i = \frac{\mu}{\sqrt{2a_i}} - \frac{\mu}{\sqrt{2a_f}} \]
- and similarly 
  \[ \Delta C_f = C_f - C_i = \frac{\mu}{\sqrt{2a_i}} \frac{a_f - a_i}{a_f + a_i} \]
- These changes are instantaneous, so \( r \) does not change, and the potential energy cannot change as well. The kinetic energy changes required are related to the velocity changes by 
  \[ \Delta v_r = \frac{1}{2} \left( v_r + \Delta v_r \right)^2 - \frac{1}{2} v_r^2 \]
- Eliminating DC we find 
  \[ \Delta v_r = \frac{\mu}{\sqrt{2} a_i} \left[ 1 + \frac{1}{a_f/a_i} \right] \]
Transfer between Orbits

Example: LEO to GEO

Example: LEO to GEO

Example: LEO to GEO

Transfer between orbits

Example: LEO to GEO
Transfer between orbits

- If the applied pulse is large enough, a parabolic or hyperbolic orbit is obtained.
- For a parabolic orbit we need to impress a velocity change
  \[ \Delta v = (\sqrt{2} - 1)v_i = (\sqrt{2} - 1)\frac{\mu}{\sqrt{a_i}} \]
- Any change larger than that will result in a hyperbolic orbit with eccentricity \( e > 1 \).
- These orbits can be used to escape the Earth gravitational field to arrive to the Moon, Mars or further away (interplanetary travel).
- A planetary fly-past can be used as a velocity amplifier, to reach farther regions of space.