Lectures 3, 4 and 5: Orbits

Reference Book:

A.E. Roy: Orbital Motion, 3rd edition 1988
Adam Hilger ed.Bristol, Philadelphia
ISBN 0-85274-229-0

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Orbital Motion

- Is the motion of a body around another one.
- Examples:
  - A solar system planet around the Sun
  - A star around a companion star (binary system)
  - A Galaxy around another (like LMC around the Milky Way)
  - A satellite around a planet
  - **An artifical satellite around the Earth**
- Basic experimental data come from observation of the orbits of the planets around the Sun.
- Kepler's laws are very good approximations of precise astronomical measurements.

Kepler's (empirical) laws

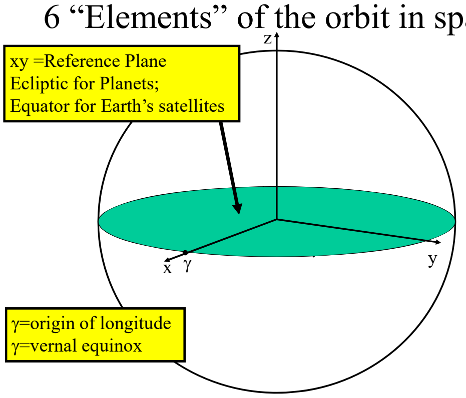
1. The orbit of each planet is an ellipse with the Sun at one focus
2. For any planet the rate of description of area by the radius vector joining the planet to Sun is constant
3. The cubes of the semimajor axes of the planetary orbits are proportional to the squares of the planets' periods of revolution.

- ❑ At the time of their formulation (circa 1600) these laws described prefectly the observational data. Even today, with extremely precise data, these laws are a close first approximation to the thruth. They also hold for various systems of satellites orbiting their primary.
- ❑ They fail only for close satellites of a nonspherical planet and for the outermost retrograde satellites.

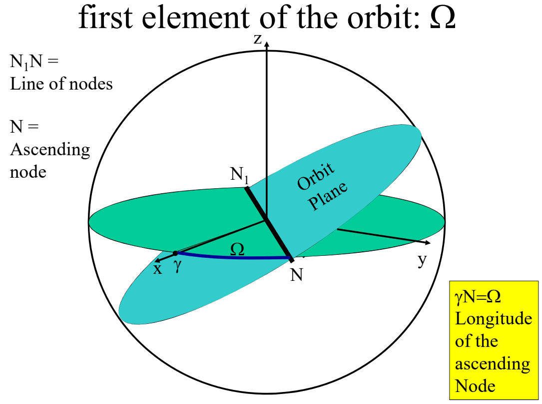
Kepler's laws explained

- Newton was the first one to explain these laws as a result of the laws of Dynamics and Gravitation.
- Kepler's laws are a description of a special case of the solution of the gravitational problem of n bodies, where
  - All bodies can be treated as point-masses
  - All the masses but one are so small that they do not attract each other appreciably, but they are attracted solely by the large mass.
- These two conditions are verified quite well for the solar system planets orbiting around the Sun.
- We are now going to study this problem. The first thing to do is to define a suitable way to define an orbit in space. 6 quantities are needed.

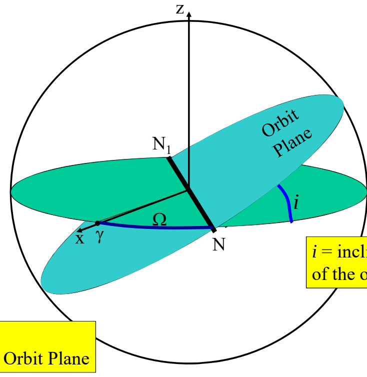
6 "Elements" of the orbit in space



first element of the orbit: Ω



second element of the orbit: i

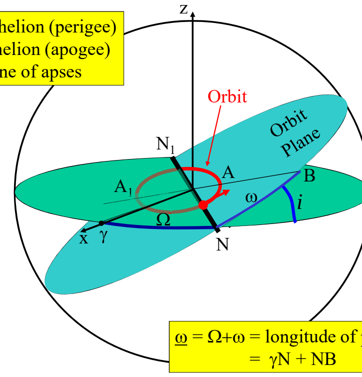


i = inclination of the orbit

i and Ω define the Orbit Plane

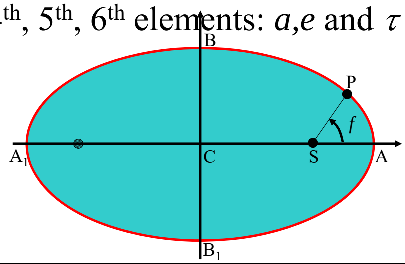
third element of the orbit: ω

A = perihelion (perigee)
A$_1$ = aphelion (apogee)
A$_1$A = line of apses



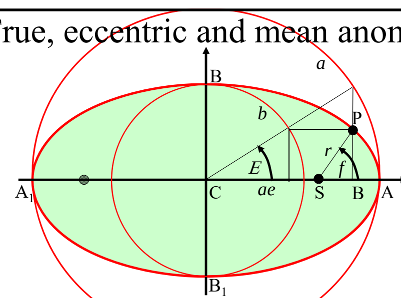
ω = Ω+ω = longitude of perihelion = = γN + NB

4th, 5th, 6th elements: a,e and τ



- a = semimajor axis: A$_1$A=2a
- e = ellipticity: CS=ae= → $e=\frac{\sqrt{a^2-b^2}}{a}$
- τ = time of perihelion (perigee) passage
- R = SP= vector radius
- f = true anomaly, often used in addition

True, eccentric and mean anomaly

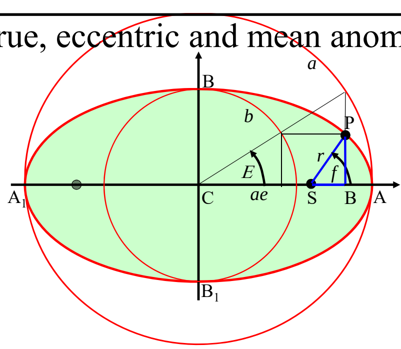


$$r = a(1 - e\cos E)$$

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

- $ACP= E$ = eccentric anomaly
- $ASP= f$ = true anomaly
- $M=2\pi(t-\tau)/T$ = mean anomaly

True, eccentric and mean anomaly



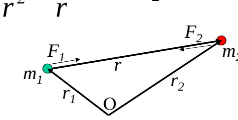
Considerando il triangolo blu :

$$r^2 = (b\cos E)^2 + (a\cos E - ae)^2 \rightarrow r = a(1 - e\cos E)$$

The two-body problem

- Given at any time the positions and velocities of two known point-masses moving under their mutual gravitational force, compute their position and velocities at any other time.
- First solved by Newton. $\frac{d}{dt}(m\vec{v}) = \vec{F}$
- Needed for the solution:

$$\vec{F}_1 = G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r} \quad ; \quad \vec{F}_2 = -G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r}$$



The two-body problem

- Motion equations:

$$m_1\frac{d^2\vec r_1}{dt^2}=G\frac{m_1m_2}{r^2}\frac{\vec r}{r}\quad;\quad m_2\frac{d^2\vec r_2}{dt^2}=-G\frac{m_1m_2}{r^2}\frac{\vec r}{r}$$

- **Motion of the centre of mass** : add eqns. above:

$$m_1\frac{d^2\vec r_1}{dt^2}+m_2\frac{d^2\vec r_2}{dt^2}=0 \qquad \vec R_G \underset{def}{=} \frac{m_1\vec r_1+m_2\vec r_2}{m_1+m_2}$$

$$m_1\frac{d\vec r_1}{dt}+m_2\frac{d\vec r_2}{dt}=\vec a$$

$$m_1\vec r_1+m_2\vec r_2=\vec a t+\vec b \Longrightarrow \boxed{(m_1+m_2)\vec R_G=\vec a t+\vec b}$$

- The centre of mass moves with constant velocity

The two-body problem

- Motion equations: $\frac{d^2\vec r_1}{dt^2}=G\frac{m_2}{r^2}\frac{\vec r}{r}\quad;\quad\frac{d^2\vec r_2}{dt^2}=-G\frac{m_1}{r^2}\frac{\vec r}{r}$
- **Relative motion** : subtract eqns. above:

$$\frac{d^2\vec r}{dt^2}+G(m_1+m_2)\frac{\vec r}{r^3}=0\ \Rightarrow\ \frac{d^2\vec r}{dt^2}+\mu\frac{\vec r}{r^3}=0$$

- Take the vector product of ***r*** with equation above:

$$\vec r\times\left(\frac{d^2\vec r}{dt^2}+\mu\frac{\vec r}{r^3}\right)=0\ \Rightarrow\ \vec r\times\frac{d^2\vec r}{dt^2}=0$$

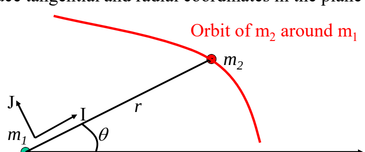
- Integrating by parts

$$\vec h=\int\vec r\times\frac{d^2\vec r}{dt^2}dt=\vec r\times\frac{d\vec r}{dt}+\int\frac{d\vec r}{dt}\times\frac{d\vec r}{dt}dt=\vec r\times\frac{d\vec r}{dt}$$

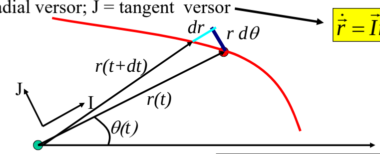
- Where ***h*** is a constant vector. The motion (orbit) lies in a plane perpendicular to ***h***. **(Angular momentum integral).**

The two-body problem

- Introduce tangential and radial coordinates in the plane of the orbit:



- I = radial versor; J = tangent versor $\boxed{\dot{\vec r}=\vec I\dot r+\vec J r\dot\theta}$



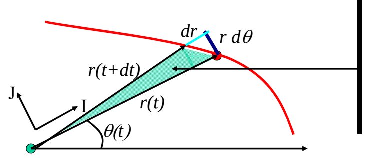
- Also: $\boxed{\ddot{\vec r}=\vec I(\ddot r-r\dot\theta^2)+\vec J\frac{1}{r}\frac{d}{dt}(r^2\dot\theta)}$

The two-body problem

- So $\vec h=\vec r\times\frac{d\vec r}{dt}$ can be rewritten $\vec h=\vec I r\times(\vec I\dot r+\vec J r\dot\theta)=\vec K r^2\dot\theta$

which means that the quantity $r^2\dot\theta=h$ is a constant.

- This is the mathematical form of **Kepler's second law**: the motion along the orbit happens with constant area velocity.



Area swapped in time dt : $dA=\frac{1}{2}r\times rd\theta$

$\boxed{\dot A=\frac{1}{2}r^2\dot\theta=\frac{1}{2}h}$

The two-body problem

- Relative motion equation : $\frac{d^2\vec r}{dt^2}+\mu\frac{\vec r}{r^3}=0$
- multiply (scalar) by $\dot{\vec r}$: $\dot{\vec r}\bullet\frac{d^2\vec r}{dt^2}+\mu\frac{\dot{\vec r}\bullet\vec r}{r^3}=0$
- Integrating: $\int\dot{\vec r}\bullet\frac{d^2\vec r}{dt^2}dt+\int\mu\frac{\dot{\vec r}\bullet\vec r}{r^3}dt=C$

$$\int\dot{\vec r}\bullet d\dot{\vec r}+\mu\int\frac{\vec I r\bullet(\vec I\dot r+\vec J r\dot\theta)}{r^3}dt=C$$

$$\frac{\dot r^2}{2}+\mu\int\frac{r\dot r}{r^3}dt=C\ \Rightarrow\ \boxed{\frac{\dot r^2}{2}-\frac{\mu}{r}=C}$$

- This is related to **energy conservation**.

The two-body problem

- Relative motion equation : $\frac{d^2\vec r}{dt^2}+\mu\frac{\vec r}{r^3}=0$
- Can be rewritten in polar coordinates:

$$\vec I(\ddot r-r\dot\theta^2)+\vec J\frac{1}{r}\frac{d}{dt}(r^2\dot\theta)+\mu\frac{\vec I}{r^2}=0$$

- So that

$$\begin{cases}\ddot r-r\dot\theta^2+\frac{\mu}{r^2}=0\\ \frac{1}{r}\frac{d}{dt}(r^2\dot\theta)=0\ \Rightarrow\ r^2\dot\theta=h\end{cases}$$

- It is useful to introduce the new variable $u=1/r$:

$$r=\frac{1}{u}\ \rightarrow\ \frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{dt}\ \rightarrow\ \frac{d^2r}{dt^2}=\frac{2}{u^3}\left(\frac{du}{dt}\right)^2-\frac{1}{u^2}\frac{d^2u}{dt^2}$$

$$\dot\theta=hu^2$$

The two-body problem

$$r=\frac{1}{u} \;\to\; \frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{dt} \;\to\; \frac{d^2r}{dt^2}=\frac{2}{u^3}\left(\frac{du}{dt}\right)^2-\frac{1}{u^2}\frac{d^2u}{dt^2}$$

$$\dot{\theta}=hu^2$$

- Substituting into $\ddot{r}-r\dot{\theta}^2+\frac{\mu}{r^2}=0$
- We get: $\frac{2}{u^3}\left(\frac{du}{dt}\right)^2-\frac{1}{u^2}\frac{d^2u}{dt^2}-\frac{1}{u}h^2u^2+\mu u^2=0$
- Now we want to eliminate t in order to get the orbit equation *u(θ)* (i.e. *r(θ)*):

$$\frac{du}{dt}=\frac{du}{d\theta}\dot{\theta}=hu^2\frac{du}{d\theta} \quad \text{and} \quad \frac{d^2u}{dt^2}=\frac{d}{dt}\left[hu^2\frac{du}{d\theta}\right]=\frac{d}{d\theta}\left[hu^2\frac{du}{d\theta}\right]\frac{d\theta}{dt}\to$$

$$\frac{d^2u}{dt^2}=\left[2u\left(\frac{du}{d\theta}\right)^2+u^2\frac{d^2u}{d\theta^2}\right]h^2u^2$$

The two-body problem

- Substituting into $\frac{2}{u^3}\left(\frac{du}{dt}\right)^2-\frac{1}{u^2}\frac{d^2u}{dt^2}-\frac{1}{u}h^2u^2+\mu u^2=0$
- We get: $\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}$
- The solution is:

$$u=\frac{\mu}{h^2}+A\cos(\theta-\omega) \;\to\; r(\theta)=\frac{h^2/\mu}{1+\left(Ah^2/\mu\right)\cos(\theta-\omega)}$$

- With A and ω constants. This is a conic section:
- So that

$$\boxed{r=\frac{p}{1+e\cos(\theta-\omega)}}$$

$$p=\frac{h^2}{\mu} \quad ; \quad e=\frac{Ah^2}{\mu}$$

$e=0$	*circle*
$0<e<1$	*ellipse*
$e=1$	*parabola*
$e>1$	*hyperbola*

Elliptic Orbit

- Let's study in detail the elliptic orbit.
- Relevant definitions / relations :

$2a = major\ axis$

$2b = minor\ axis$

$CS/CA = e$

$b=a\sqrt{1-e^2}$

$SP+S'P=2a$

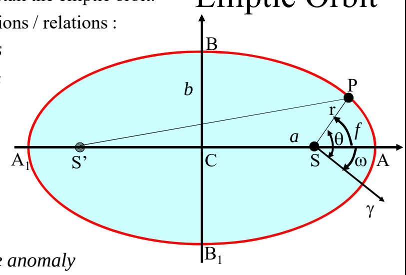
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$f=\theta-\omega \quad true\ anomaly$

$r=\frac{p}{1+e\cos(\theta-\omega)}$

$\theta=\omega \quad perihelion$

$\theta=\pi+\omega \quad aphelion$



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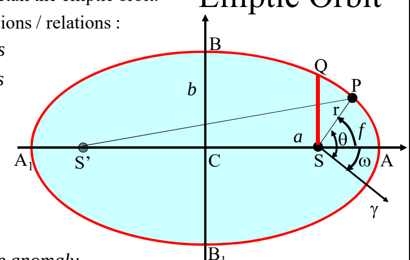


$QS=a(1-e^2)$

$QS=r(\pi/2)=p$

$r=\frac{a(1-e^2)}{1+e\cos(\theta-\omega)}$

Elliptic Orbit

- The area of the ellipse is $A=\pi ab$.
- The second law of Kepler can be written

$$h=2\frac{dA}{dt}\to \quad h\int dt=2\int dA \quad \to hT=2\pi ab \quad \to \frac{2\pi ab}{T}=h$$

- Moreover, the solution of the two-body problem gives $a(1-e^2)=p=\frac{h^2}{\mu} \quad \to h=\sqrt{\mu a}(1-e^2)^{1/2}$
- So combining the two equations we have:

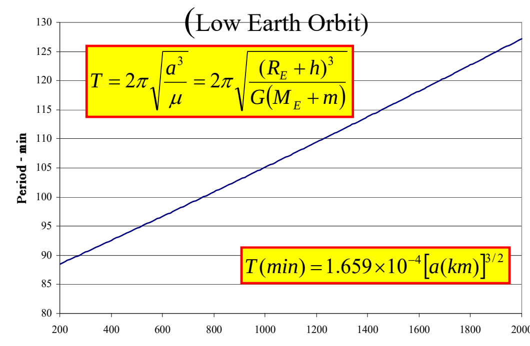
$$\frac{2\pi ab}{T}=\sqrt{\mu a}(1-e^2)^{1/2} \quad \to \quad \frac{2\pi a^2\cancel{(1-e^2)^{1/2}}}{T}=\sqrt{\mu a}\cancel{(1-e^2)^{1/2}}$$

$$\boxed{\to \quad T=2\pi\sqrt{\frac{a^3}{\mu}}}$$

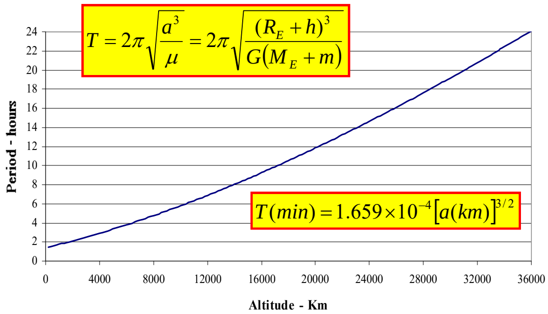
The period depends ONLY on the semimajor axis and on the sum of the two masses

LEO satellite period

(Low Earth Orbit)



Satellite period for higher orbits



Elliptic Orbit

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

- As an immediate consequence we have that for two masses m_1 and m_2 orbiting around the same centre mass M_s

$$\frac{M_s + m_1}{M_s + m_2} = \left(\frac{a_1}{a_2}\right)^3 \left(\frac{T_2}{T_1}\right)^2$$

- This is the correct form of the third law of Kepler for the Solar System. Here

$$M_s >> m_i \quad \rightarrow \quad \left(\frac{a_1}{a_2}\right)^3 \left(\frac{T_2}{T_1}\right)^2 \cong 1$$

Nicholas Copernicus (1473-1543)

In *De Revolutionibus Orbium Coelestium* ("On the Revolutions of the Celestial Orbs"), which was published in Nuremberg in 1543, the year of his death, stated that that the Sun was the center of the universe and that the Earth had a triple motion around this center.

His theory gave a simple and elegant explanation of the retrograde motions of the planets (the annual motion of the Earth necessarily projected onto the motions of the planets in geocentric astronomy) and settled the order of the planets (which had been a convention in Ptolemy's work) definitively.

Tycho Brahe (1546-1601)

•Tycho designed and built new instruments, calibrated them, and instituted nightly observations.

•Changed observational practice profoundly: earlier astronomers observed the positions of planets and the Moon at certain important points of their orbits (e.g., opposition, quadrature, station), Tycho observed these bodies throughout their orbits.

•As a result, a number of orbital anomalies never before noticed were made explicit by Tycho. Without these complete series of observations of unprecedented accuracy, Kepler could not have discovered that planets move in elliptical orbits.

Johannes Kepler (1571-1630)

•Using the precise data that Tycho had collected, Kepler discovered that the orbit of Mars was an ellipse.

•In 1609 he published *Astronomia Nova*, delineating his discoveries, which are now called Kepler's first two laws of planetary motion. In 1619 he published *Harmonices Mundi*, in which he describes his "third law."

•Kepler published the seven-volume *Epitome Astronomiae* in 1621. This was his most influential work and discussed all of heliocentric astronomy in a systematic way. He was a sustainer of the copernican system.

Velocity in an Elliptic Orbit

- Let V be the velocity of the orbiting mass in the position r,f.
- Since $\vec{r} = \vec{I}\dot{r} + \vec{J}r\dot{\theta}$
- The modulus of the velocity is thus $V^2 = \dot{r}^2 + r^2\dot{f}^2$
- It is possible to express this as a function of r only. We use

$$r = \frac{p}{1+e\cos f} \quad \rightarrow \dot{r} = \frac{pe\sin f}{(1+e\cos f)^2}\dot{f} = \frac{r^2}{p^2}pe\sin f\,\frac{h}{r^2} = \frac{he}{p}\sin f \qquad (1)$$

$$r^2\dot{f} = h \quad \rightarrow r\dot{f} = \frac{h}{r} = \frac{h}{p}(1+e\cos f) \qquad (2)$$

- Inserting (1) and (2) inside $V^2 = \dot{r}^2 + r^2\dot{f}^2$ we get:

$$V^2 = \left(\frac{he}{p}\sin f\right)^2 + \left(\frac{h}{p}(1+e\cos f)\right)^2 = \left(\frac{h}{p}\right)^2\left(e^2\sin^2 f + 1 + 2e\cos f + e^2\cos^2 f\right) =$$

$$= \left(\frac{h}{p}\right)^2\left(e^2 + 2e\cos f + 1\right) = \left(\frac{h}{p}\right)^2\left(e^2 + 2e\cos f + 1\right) = \left(\frac{h}{p}\right)^2\left(2 + 2e\cos f - (1-e^2)\right) =$$

Velocity in an Elliptic Orbit

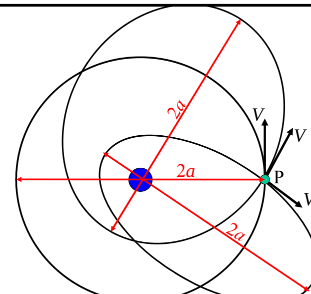
$$V^2=\left(\frac{h}{p}\right)^2\left(2+2e\cos f-(1-e^2)\right)=\left(\frac{h}{p}\right)^2\left[2(1+e\cos f)-(1-e^2)\right]=$$

$$=\frac{2h^2}{rp}-\left(\frac{h}{p}\right)^2(1-e^2)=\frac{2\mu}{r}-\mu\frac{a(1-e^2)(1-e^2)}{a^2(1-e^2)^2}\;\to$$

$$\boxed{V^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)}$$

- This formula shows that the velocity depends only on the radius r and allows the computation of the velocity in any point of the orbit :
  - At pericenter $r=a(1-e)\;\to\;V_P^2=\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)$
  - At apocenter $r=a(1+e)\;\to\;V_A^2=\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)$
- If a satellite is injected at distance r from the center body and with initial velocity V, the semimajor axis of the orbit *a* will not depend on the direction of the inital velocity.

$$\boxed{V^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)}$$



- If a satellite is injected at distance *r* from the center body and with initial velocity *V*, the semimajor axis of the orbit *a* will not depend on the direction of the inital velocity.
- Moreover, for all the orbits with different velocity directions and same *V* and *r*, the period *T* is also the same, because *T* depends on *a* only. So if 3 satellites are injected from P along the 3 orbits, they will be back in P at the same time.

$$T=2\pi\sqrt{\frac{a^3}{\mu}}$$

Velocity in an Circular Orbit

$$V^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)$$

- If the body has to move on a circular orbit, *r=constant=a*, so that

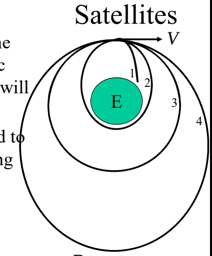
$$V_{circ}^2=\frac{\mu}{a}$$

- Note that this is half of the V_{escape}^2 we found in the previous lesson:

$$V_{escape}=V_{circ}\sqrt{2}$$

Elliptic Orbits of Artificial Satellites

- Artificial satellites are carried in orbit by multi-stage rockets. At the separation point, at an altitude *h*, the last stage of the rocket impresses to the satellite a velocity *V*.
- In the figure we see what happens if the velocity at separation is orthogonal to the radius vector. If the velocity is too small, the satellite will fall to the ground on an elliptic section (1). Otherwise the separation point will be the apogee (2).
- The minimum velocity at separation needed to obtain a full orbit can be computed imposing that at apogee $r_A=a(1+e)=R_E+h$
- And at perigee $r_P=a(1-e)=R_E$
- Moreover, for such an orbit $2a=h+2R_E$
- So the velocity at separation has to be $V_A^2=\frac{\mu}{a}\frac{(1-e)}{(1+e)}=\frac{\mu}{R_E+h/2}\frac{R_E}{R_E+h}$



Elliptic Orbits of Artificial Satellites

$$V>\sqrt{\frac{\mu}{R_E}}\sqrt{\frac{R_E}{R_E+h/2}}\sqrt{\frac{R_E}{R_E+h}}$$

- For h=150km, V> 7.77 km/s.
- Increasing the velocity the orbit will become circular for

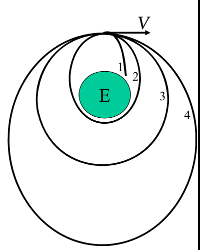
$$V^2=\sqrt{\frac{\mu}{R_E}}\sqrt{\frac{R_E}{R_E+h}}$$

(curve 3, V= 7.82 km/s), and then will be elliptical again, but the separation point will become the perigee (4).

- Further increasing, at the escape velocity

$$V^2=\sqrt{\frac{\mu}{R_E}}\sqrt{\frac{2R_E}{R_E+h}}$$

(V= 11.1 km/s) the orbit will become parabolic, and then hyperbolic (see later).



Angle between velocity and radius vector

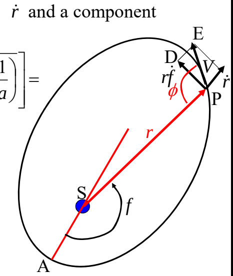
- The angle ϕ between the velocity vector and the radius vector is SPE.
- But *SPE=90º+DPE*
- The velocity *V* has a radial component $\dot r$ and a component orthogonal to the radius vector $r\dot f$, so

$$\cos DPE=\frac{DP}{PE}=\frac{r\dot f}{V}=\frac{h}{rV}=h\Big/\left[r\sqrt{\mu\left(\frac{2}{r}-\frac{1}{a}\right)}\right]=$$

$$=\sqrt{a(1-e^2)}\Big/\left[r\sqrt{\mu\left(\frac{2}{r}-\frac{1}{a}\right)}\right]=\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}$$

- So $\varphi=SPE=90^o+\cos^{-1}\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}\;\to$

$$\boxed{\sin\varphi=\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}}$$



Kepler's equation

- We first derive the relationship between the true anomaly f and eccentric anomaly E:

$$SR = CR - CS = a\cos E - ae$$

- But

$$SR = r\cos f \rightarrow$$

$$r\cos f = a(\cos E - e) \qquad (1)$$

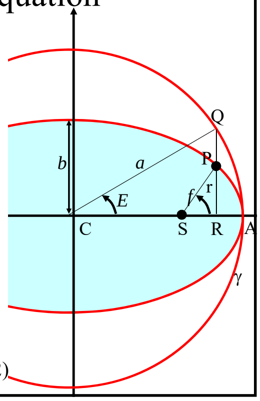
- A property of ellipses and eccentric circles is that

$$PR/QR = b/a$$

- so

$$\frac{r\sin f}{a\sin E} = \frac{b}{a} \quad \rightarrow$$

$$r\sin f = b\sin E = a\sqrt{1-e^2}\,\sin E \qquad (2)$$



Kepler's equation

- Combining (1) and (2)

$$r^2(\cos^2 f + \sin^2 f) = r^2 =$$

$$= (a\cos E - ae)^2 + a^2(1-e^2)\sin^2 E =$$

$$= a^2(1-e\cos E)^2 \quad \rightarrow$$

$$r = a(1-e\cos E)$$

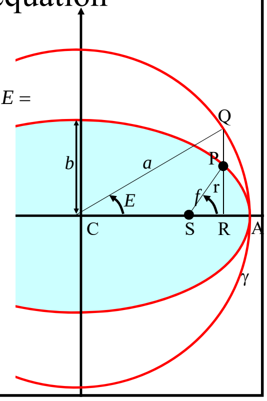
- Moreover

$$r\cos f = r\left(1-2\sin^2\frac{f}{2}\right) \quad \rightarrow$$

$$2r\sin^2\frac{f}{2} = r(1-\cos f) =$$

$$= a(1-e\cos E) - a(\cos E - e) =$$

$$= a(1+e)(1-\cos E)$$



Kepler's equation

$$2r\sin^2\frac{f}{2} = a(1+e)(1-\cos E)$$

- And similarily

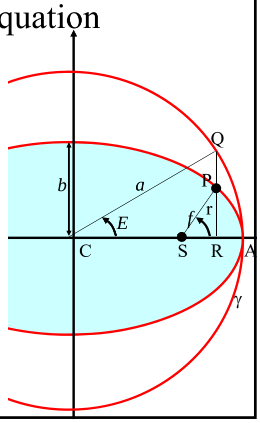
$$2r\cos^2\frac{f}{2} = a(1-e)(1+\cos E)$$

- So we get:

$$\tan^2\frac{f}{2} = \frac{a(1+e)(1-\cos E)}{a(1-e)(1+\cos E)} \quad \rightarrow$$

$$\boxed{\tan\frac{f}{2} = \sqrt{\frac{(1+e)}{(1-e)}}\,\tan\frac{E}{2}}$$

- which is the relation between eccentric anomaly E and true anomaly *f*.



Kepler's equation

- Now we derive Kepler's equation, which relates the mean anomaly M and the eccentric anomaly E .
- By Kepler's second law:

$$\frac{area\ SPA}{\pi ab} = \frac{t-\tau}{T}$$

- Or

$$area\ SPA = \frac{M}{2}ab$$

- Now

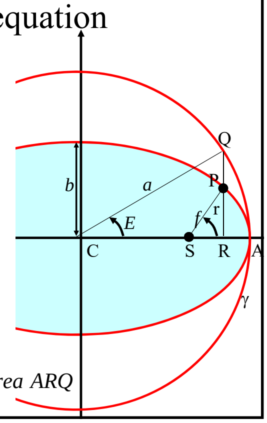
$$area\ SPA =$$

$$= area\ SPR + area\ ARP$$

- But

$$area\ ARP = \frac{a}{b}area\ ARQ$$

- so $area\ SPA = area\ SPR + \frac{a}{b}area\ ARQ$



Kepler's equation

$$areaSPA = areaSPR + \frac{a}{b}areaARQ =$$

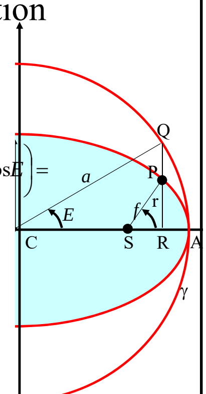
$$= areaSPR + \frac{a}{b}(areaACQ - areaCRQ) =$$

$$= \frac{1}{2}r\sin f\,r\cos f + \frac{a}{b}\left(\pi a^2\frac{E}{2\pi} - \frac{1}{2}a\sin E\,a\cos E\right) =$$

$$= \frac{1}{2}a\sqrt{1-e^2}\,\sin E\,a(\cos E - e) +$$

$$+ \frac{b}{a}\left(\frac{a^2E}{2} - \frac{1}{2}a\sin E\,a\cos E\right) =$$

$$= \frac{1}{2}ab(E - e\sin E)$$



Kepler's equation

- Combining

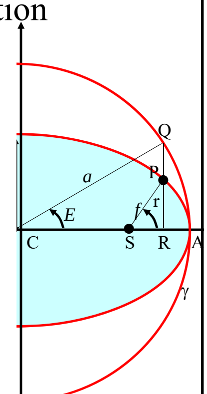
$$areaSPA = \frac{1}{2}ab(E - e\sin E)$$

- and

$$area\ SPA = \frac{M}{2}ab$$

- We finally get Kepler's equation:

$$\boxed{2\pi\frac{t-\tau}{T} = M = E - e\sin E}$$



- Now we have everything in place to solve the most common elliptic orbit problems, which are:
  - Get the position and velocity of the orbiting body, given the elements and the time
  - Obtain the elements of the orbit, given the position and velocity and the time
- The needed formulae are summarized here :

$$r=\frac{a(1-e^2)}{1+e\cos f} \quad (1) \qquad h^2=\mu a(1-e^2) \quad (6)$$
$$r=a(1-e\cos E) \quad (2) \qquad V^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right) \quad (7)$$
$$\tan\frac{f}{2}=\sqrt{\frac{(1+e)}{(1-e)}}\tan\frac{E}{2} \quad (3) \qquad T=2\pi/n \quad (8)$$
$$M=E-e\sin E \quad (4) \qquad n=\mu^{1/2}a^{-3/2} \quad (9)$$
$$M=n(t-\tau) \quad (5) \qquad \sin\varphi=\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}} \quad (10)$$

- For example, given the time t, the elements a, e, τ , and assuming μ is known, the position and velocity of the body can be found as follows:
- Compute n form (9), then find M from (5), then E from (4), then r from (2), then f from (3), then compute V from (7), then ϕ from (10).

$$r=\frac{a(1-e^2)}{1+e\cos f} \quad (1) \qquad h^2=\mu a(1-e^2) \quad (6)$$
$$r=a(1-e\cos E) \quad (2) \qquad V^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right) \quad (7)$$
$$\tan\frac{f}{2}=\sqrt{\frac{(1+e)}{(1-e)}}\tan\frac{E}{2} \quad (3) \qquad T=2\pi/n \quad (8)$$
$$M=E-e\sin E \quad (4) \qquad n=\mu^{1/2}a^{-3/2} \quad (9)$$
$$M=n(t-\tau) \quad (5) \qquad \sin\varphi=\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}} \quad (10)$$

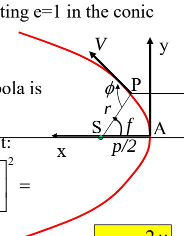
- Conversely, given V, r, ϕ, t and assuming that μ is known, the elements of the orbit can be found as follows:
- Compute a from (7), then find e from (10), then E from (2), then f from (1) (or from (3)), then compute M from (4), then τ from (5).

$$r=\frac{a(1-e^2)}{1+e\cos f} \quad (1) \qquad h^2=\mu a(1-e^2) \quad (6)$$
$$r=a(1-e\cos E) \quad (2) \qquad V^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right) \quad (7)$$
$$\tan\frac{f}{2}=\sqrt{\frac{(1+e)}{(1-e)}}\tan\frac{E}{2} \quad (3) \qquad T=2\pi/n \quad (8)$$
$$M=E-e\sin E \quad (4) \qquad n=\mu^{1/2}a^{-3/2} \quad (9)$$
$$M=n(t-\tau) \quad (5) \qquad \sin\varphi=\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}} \quad (10)$$

Parabolic Orbit

- In this type of two-body motion the orbit is open. The second body apporaches the first one from infinity and, after the nearest approach, recedes back to infinity.
- The equation of the orbit is obtained by putting e=1 in the conic equation: $r=\frac{p}{1+\cos f}=\frac{h^2/\mu}{1+\cos f}$
- In the Axy system the equation of the parabola is $y^2=2px$
- The velocity is found as for the elliptic orbit:

$$V^2=\dot r^2+r^2\dot f^2=\left[\frac{h}{p}\sin f\right]^2+\left[\frac{h}{p}(1+\cos f)\right]^2=$$
$$=\left[\frac{h}{p}\right]^2 2(1+\cos f)=\left[\frac{h}{p}\right]^2 2\frac{p}{r}=\frac{2h^2}{pr}=\frac{2\mu}{r}$$



$$V_{par}^2=\frac{2\mu}{r}$$

Circular and parabolic velocity

- Consider a satellite orbiting in a circular orbit with velocity $V_{circ}^2=\frac{\mu}{r}$
- Imagine the satellite is given an impulse so that it reaches the parabolic velocity in the same point:

$$V_{par}^2=\frac{2\mu}{r}$$

- The satellite will now follow a parabolic orbit that will take it to infinity.It will reach inifinity with 0 velocity, so the parabolic velocity is the same as the escape velocity.
- If the impulse is larger, the satellite will go to infinity on a hyperbolic orbit.

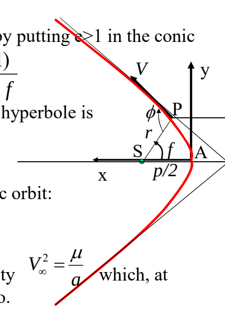
Hyperbolic Orbit

- This orbit is also open. The second body apporaches the first one from infinity and, after the nearest approach, recedes back to infinity.
- The equation of the orbit is obtained by putting e>1 in the conic equation: $r=\frac{p}{1+e\cos f}=\frac{a(e^2-1)}{1+e\cos f}$
- In the Axy system the equation of the hyperbole is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad ; \quad b^2=a^2(e^2-1)$$

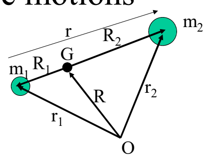
- The velocity is found as for the elliptic orbit:

$$V^2=\mu\left[\frac{2}{r}+\frac{1}{a}\right]$$



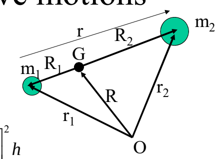
- At pericenter $V_P^2=\frac{\mu}{a}\left[\frac{e+1}{e-1}\right]$; at infinity $V_\infty^2=\frac{\mu}{a}$ which, at variance with the parabola, is non zero.

Absolute and relative motions



- If G is centre of mass of the two masses m_1 and m_2, its position vector R is
$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 \quad ; \quad M = m_1 + m_2$$
- If we put the origin O in G we see that
$$m_1R_1 = m_2R_2 \quad ; \quad r = R_1 + R_2$$
- Where r is the position of m_2 with respect to m_1. From the two equations above we get immediately
$$R_1 = \frac{m_2}{M} r \quad ; \quad R_2 = \frac{m_1}{M} r$$
- For the relative motion the second law of Kepler applies: $r^2\dot{f} = h$, and since the two masses and the center of mass are always on a straight line, the same law must apply for the barycentric orbits as well: $R_1^2\dot{f} = h_1 \quad ; \quad R_2^2\dot{f} = h_2$

Absolute and relative motions



- From $R_1 = \frac{m_2}{M} r \quad ; \quad R_2 = \frac{m_1}{M} r$
and $R_1^2\dot{f} = h_1 \quad ; \quad R_2^2\dot{f} = h_2$
we get
$$h_1 = \left[\frac{m_2}{M}\right]^2 r^2\dot{f} = \left[\frac{m_2}{M}\right]^2 h \quad ; \quad h_2 = \left[\frac{m_1}{M}\right]^2 h$$
- The barycentric orbits are thus geometrically similar to each other and to their relative orbit.
- This means that they have same eccentricities and same periods, and that, for example
$$a_1 = \frac{m_2}{M} a \quad ; \quad a_2 = \frac{m_1}{M} a$$

Orbits and Energy

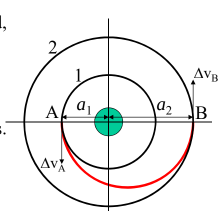
- We are now in a position to relate the shape of the orbits to the total energy of the system.
- If V1 and V2 are the velocities with respect to the center of mass (taken to be at rest), the total energy of the system is
$$E = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 - G\frac{m_1m_2}{r}$$
- Now $V_1^2 = \dot{R}_1^2 + R_1^2\dot{f}_1^2 = \left[\frac{m_2}{M}\right]^2\left(\dot{r}^2 + r^2\dot{f}^2\right) = \left[\frac{m_2}{M}\right]^2 V^2 \quad ; \quad V_2^2 = \left[\frac{m_1}{M}\right]^2 V^2$
- So we get
$$E = \frac{1}{2}m_1\left[\frac{m_2}{M}\right]^2 V^2 + \frac{1}{2}m_2\left[\frac{m_1}{M}\right]^2 V^2 - G\frac{m_1m_2}{r} = \frac{m_1m_2}{M}\frac{1}{2}V^2 - G\frac{m_1m_2}{r} =$$
$$= \frac{m_1m_2}{m_1+m_2}\left[\frac{1}{2}V^2 - G\frac{m_1+m_2}{r}\right] = \frac{m_1m_2}{m_1+m_2}\left[\frac{1}{2}V^2 - \frac{\mu}{r}\right]$$
- The quantity in squares has been shown to be a constant of the motion, called *C*. If the mass of the satellite m_1 is much smaller than the mass of the planet m_2 , we get $E \cong m_1\left[\frac{1}{2}V^2 - \frac{\mu}{r}\right] = m_1C$
- So C is the total energy of the satellite per unit mass.

Orbits and Energy

- C is different for the different orbits:
- For the ellipse
$$C = \frac{1}{2}V^2 - \frac{\mu}{r} = \frac{1}{2}\mu\left[\frac{2}{r} - \frac{1}{a}\right] - \frac{\mu}{r} = -\frac{\mu}{2a} < 0$$
- For the parabola
$$C = \frac{1}{2}V^2 - \frac{\mu}{r} = \frac{1}{2}\frac{2\mu}{r} - \frac{\mu}{r} = 0$$
- For the hyperbola
$$C = \frac{1}{2}V^2 - \frac{\mu}{r} = \frac{1}{2}\mu\left[\frac{2}{r} + \frac{1}{a}\right] - \frac{\mu}{r} = \frac{\mu}{2a} > 0$$
- Hence for a closed orbit the total energy (kinetic plus potential) must be negative, for escape just to take place the total energy is zero; for an energy >0 an escape along a hyperbola takes place.

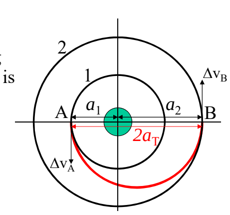
Transfer between Orbits

- If motors are not used, the satellite will follow its conic orbit, for example a circle around the Earth.
- Firing a motor will cause changes in the orbit, affecting in general all 6 elements of the orbit.
- Here we assume that firing is short enough that only the velocity is changed, while the position is not: so we assume that the impulse is applied instantly to the satellite.
- We start assuming that we want to transfer between coplanar circular orbits.
- To do this we apply an impulse in A, producing a velocity variation Δv_A, and starting a transfer orbit (red); in B we apply another impulse, producing a velocity variation Δv_B, and inserting the satellite in the final orbit (radius a_2).



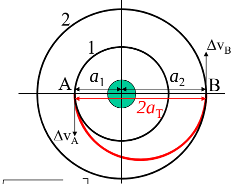
Transfer between Orbits

- It useful to treat the problem as a problem of change of energy.
- The vehiclès energies in the two orbits are $C_1 = -\frac{\mu}{2a_1} \quad ; \quad C_2 = -\frac{\mu}{2a_2}$
- So the energy required to effect the transfer is at least $\Delta C = \frac{\mu}{2}\left[\frac{1}{a_1} - \frac{1}{a_2}\right]$
- The two impulses can be applied firing the motor tangentially to the orbit: this is most economic in fuel for a given change in kinetic energy.
- The energy in a transfer ellipse is $C_T = -\frac{\mu}{2a_T}$
- But the semimajor axis of the ellipse is
$$2a_T = a_1 + a_2 \quad \rightarrow \quad C_T = -\frac{\mu}{a_1 + a_2}$$



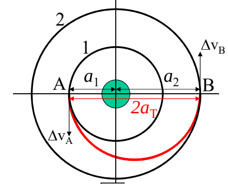
Transfer between Orbits

- So the energy required in A to place the vehicle in the correct elliptical transfer orbit is
  $$\Delta C_A = C_T - C_1 = -\frac{\mu}{a_1+a_2} + \frac{\mu}{2a_1} = \frac{\mu}{2a_1}\left[\frac{a_2-a_1}{a_2+a_1}\right]$$
- And similarly
  $$\Delta C_B = C_2 - C_T = \frac{\mu}{2a_2}\left[\frac{a_2-a_1}{a_2+a_1}\right]$$
- These changes are istantaneous, so r does not change, and the potential energy cannot change as well. The kinetic energy changes required are related to the velocity changes by
  $$\Delta C_A = \frac{1}{2}(v_A+\Delta v_A)^2 - \frac{1}{2}v_A^2$$
  $$\Delta C_B = \frac{1}{2}(v_B+\Delta v_B)^2 - \frac{1}{2}v_B^2$$
- Eliminating DC we find
  $$\Delta v_A = \sqrt{\frac{\mu}{a_1}}\left[\sqrt{\frac{2a_2/a_1}{1+a_2/a_1}} - 1\right]$$
  $$\Delta v_B = \sqrt{\frac{\mu}{a_2}}\left[1 - \sqrt{\frac{2}{1+a_2/a_1}}\right]$$



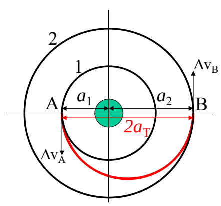
Transfer between Orbits

- The total velocity increment is
  $$\Delta v = \Delta v_A + \Delta v_B$$
- And the fuel required can be computed from the usual formula
  $$\Delta v = v_e \ln\frac{m_o}{m_B}$$
- The eccentricity of the elliptic transfer orbit is obtained from
  $$a_1 = a_T(1-e) \quad ; \quad a_2 = a_T(1+e)$$
  since the ellipse focus is on the planet.
- So
  $$e = \frac{a_2-a_1}{a_2+a_1}$$
- The time required to make the transfer is half of the period of the elliptic orbit: $t_T = \frac{T}{2} = \pi\sqrt{\frac{a_T^3}{\mu}}$
- The fuel-efficient transfer described here is called ***Hohmann transfer*.**



Example: LEO to GEO

- The Shuttle typically deploys a satellite into an orbit with an altitude of about 300 km (LEO: low Earth orbit) and an inclination of 28.5 degrees.
- To be in a geosynchronous orbit, the satellite must move to an equatorial orbit (inclination = 0 degrees) with an altitude of 38,000+ km (GEO: Geosynchrounous Orbit). This means changing two separate aspects of the initial deployment orbit, radius (or altitude) and inclination.
- The altitude change from 300 to 38000 km can be obtained with a Hohmann transfer.



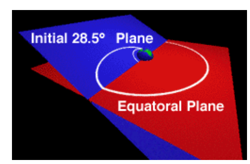
a_1=300 km

a_2=38000 km

But the two orbits are not co-planar !

Example: LEO to GEO

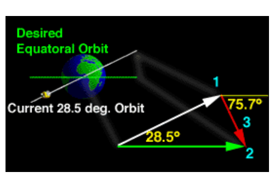
- A plane change requires an engine firing in the out-of-plane direction.
- Since the point in an orbit where the engines are fired automatically becomes a point in the new orbit (or the burn point becomes the intersection of the old and new orbits), this firing must occur where the current orbit and the desired orbit intersect.
- So, if the satellite is in an orbit inclined 28.5 degrees to the equator the firing must occur at one of two points during each orbit revolution where the spacecraft is directly over the equator.



- In this way we place the satellite in an elliptical transfer orbit bringing it at 38000 km exactly when the transfer orbit intersects the equatorial plane again.
- Here we must fire the motor off-plane.

Example: LEO to GEO

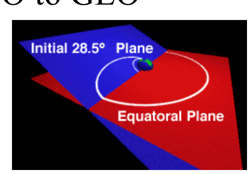
•The off-axis direction can be computed simply as shown in fig.:
•Velocity in inclined orbit: 7.726 km/sec at 28.5 degrees to equator.
•Velocity in equatoral orbit 7.726 km/sec at 0 degrees (Note: Same speed since only the angle is changing.
•Change in velocity (Delta V) needed is (1) + (3) = (2) so (3) = 3.801 km/sec at –75.7 degrees.



•And finally we need another firing to arrive to the velocity needed for the circular geostationary orbit.

Example: LEO to GEO

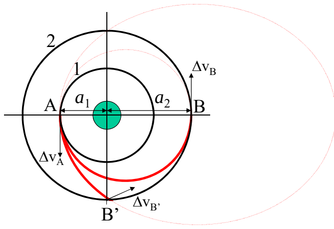
- Of course there are different ways to obtain the same final orbit: the computation of the most fuel and time effective orbit requires a significant effort.
- It turns out that the optimum solution dictates that the plane be shifted by about 2 deg. at the lower altitude and the remaining 26 deg. at the geostationary altitude.
- The only consideration left is one of precise timing. Geostationary satellites have a particular destination in orbit, a certain longitude on the equator that it is to "hover" over.



- Therefore, the scheme that we have arrived at has to occur such that when the satellite reaches the desired altitude it is above the right point on the equator.
- This involves picking the right equator crossing point of the satellite at which to initiate the transfer.

Transfer between orbits

- Transfer faster than the Hohmann one can be obtained with a stronger pulse in A as shown in the figure below: the orbit is AB’.
- The cost of this is that in B’ it will be needed to produce a second pulse which is not tangential to the destination orbit:



- This will require a larger amount of fuel, so a tradeoff between transfer time and fuel consumption has to be decided.

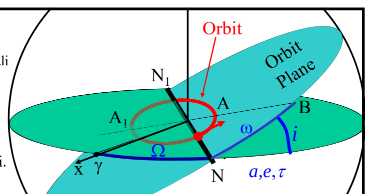
Transfer between orbits

- If the applied pulse is large enough, a parabolic or hyperbolic orbit is obtained.
- For a parabolic orbit we need to impress a velocity change

$$\Delta v = (\sqrt{2}-1)v_c = (\sqrt{2}-1)\sqrt{\frac{\mu}{a_1}}$$

- Any change larger than that will result in a hyperbolic orbit with eccentricity e>1.
- These orbits can be used to escape the Earth gravitational field to arrive to the Moon, Mars or further away (interplanetary travel).
- A planetary fly-past can be used as a velocity amplifier, to reach farther regions of space.

- L’altra volta:
  - Orbite e parametri orbitali
  - Problema dei due corpi
  - Orbite ed energia
  - Trasferimenti orbitali
- Oggi: problema dei molti corpi e argomenti correlati.



$$r = \frac{a(1-e^2)}{1+e\cos f} \quad (1) \qquad h^2 = \mu a(1-e^2) \quad (6)$$

$$r = a(1-e\cos E) \quad (2) \qquad V^2 = \mu\left(\frac{2}{r}-\frac{1}{a}\right) \quad (7)$$

$$\tan\frac{f}{2} = \sqrt{\frac{(1+e)}{(1-e)}}\tan\frac{E}{2} \quad (3) \qquad T = 2\pi / n \quad (8)$$

$$M = E - e\sin E \quad (4) \qquad n = \mu^{1/2}a^{-3/2} \quad (9)$$

$$M = n(t-\tau) \quad (5) \qquad \sin\varphi = \sqrt{\frac{a^2(1-e^2)}{r(2a-r)}} \quad (10)$$

The Many-Body problem

- Given at any time the positions and velocities of **three or more** massive particles, moving under their mutual gravitational forces, the masses also being known, calculate their positions and velocities at any other time.
- This problem is orders of magnitude more difficult than the two-body problem we have considered.
- If the **bodies** are **not point-like** and **not spherical**, additional complexity is added.
- No general analytical solution has been found in the last three centuries, despite of the enormous amound of work carried out by people like Euler, Lagrange, Poincare, etc.

The Many-Body problem

- There are 10 known integrals of the motion, which were already known to Euler: since then, no further integrals have been discovered.
- Solution in particular cases has been found by Lagrange, which are of interest in astrodynamics and in astronomy.
- Families of periodic orbits of a small test particle in the field of two masses orbiting in undisturbed circular orbits have been found at the epoch of Poincarè. These are of interest as an approximation of a space vehicle in Earth-Moon space.

The Many-Body problem

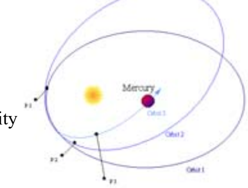
- Since the orbits of the planets in the solar system are very accurately elliptical, the solar system can be first approximated neglecting the planet-planet gravitational attraction: in this way the problem is reduced to a set of two-body problems, one for each planet.
- A better approximation is obtained assuming that the effect of planet-planet gravitational attraction is a **perturbation of the elements** of the elliptical orbits.
- Analytical expressions, valid for a period of time, have been found in this way. These are called **“general perturbations”.**
- Effects like the advance of perihelium or the retroregression of the ascending node of a planetary orbit have been found in this way.

The Many-Body problem

- The other approach to the solution is the numerical integration of the differential equations of motion. This can be done using high speed computers, and is called “special perturbations”.
- Perturbations may be divided in two further classes: **Periodic** and **Secular**.
- Any disturbance of the reference orbit that is repeated with a given period of revolution is termed a periodic perturbation.

Advance of Perihelion

- A secular perturbation, instead, causes a change proportional to time. Typical example: the advance of perihelion.
- In the case of Mercury, the advance (greatly exaggerated in the figure) is 43 arcsec/year.
- Only part of it is explained by newtonian dynamics. General relativity is needed to fully explain it.
- In the case of the binary pulsar PSR-1913+16 (the Hulse-Taylor pulsar, used to demonstrate the existence of gravitational waves) the advance is as large as 4.2°/year



- *In many cases it is difficult to distinguish between (long) periodic and secular, since the time over which perturbation measurements are available is short compared to the suspected perturbation period.*

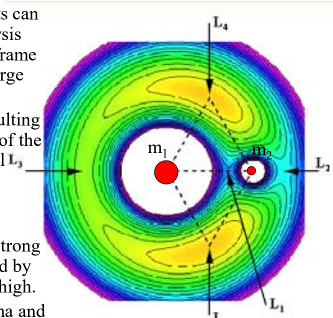
The Many-Body problem

- Finally, we should distinguish between few-body problem and many-body problems.
- Example of few-body: solar system – orbits have to be calculated precisely and the number of masses is not large enough to enable statistical or hydrodynamical approaches.
- Example of many-body: a stellar system (globular cluster, or galaxy) where the number of stars is large enough, and statistical methods are used.
- *We will not study in detail the many-body problem, but analyze a few cases of immediate interest for space astronomy methods.*

The Lagrange Points

- A very *particular case of the 3-body problem* has been studied by Joseph-Louis Lagrange in 1772 (*Essai sur le problème des trois corps*).
- This is the “*Circular Restricted Three-Body problem*”, where two massive particles move in circles about their centre of mass and attract (but are not attracted by) a third test particle of infinitesimal mass.
- It turns out that there are 5 positions where the test particle can be placed where the gravitational pull of the two large masses precisely cancels the centripetal acceleration required to rotate with them.
- These points are called “Lagrange Points L1..L5”, or “*libration points*”, because there a light third body can sit "motionless" relative to the two heavier bodies that are orbiting each other.

- The location of the 5 points can be obtained from an analysis carried out in a reference frame which rotates with the 2 large masses.
- An effective potential, resulting from the action of gravity of the two masses and centrifugal force, can be derived. It is represented in the figure.
- Near the two masses, the gravitational attraction is strong and cannot be compensated by the centrifugal force. U is high.
- The arrows point to maxima and saddles in the potential U: In these regions the two forces and the centrifugal force tend to compensate.



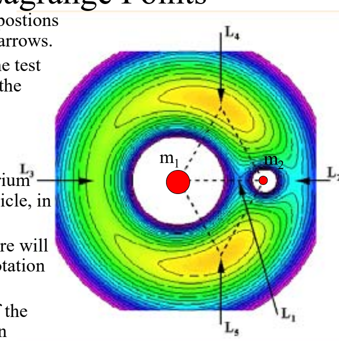
- Far from the two masses, the gravitational pull of m_1 and m_2 is not enough to compensate the centrifugal force $m\omega^2 r$. U is high.

The Lagrange Points

- So the only equilibrium postions are those pointed by the arrows.
- Here the total force on the test mass is vanishing, since the gradient of U is zero:

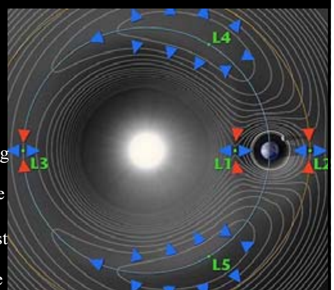
$$\vec{F} = \nabla U$$

- Points $L_1..L_5$ are equilibrium positions for the test particle, in this reference frame.
- A test particle placed there will just follow the general rotation of m_1 and m_2.
- However, the stability of the equilibrium is different in L_1,L_2,L_3 and in L_4,L_5 .



Stability

- Since the L_i points are either maxima or saddles of the effective potential, they should be unstable equilibrium points.
- However, such potential is neglecting the effect of Coriolis force. This is correct as long as the test mass is at rest in the rotating system, but is not correct anymore if we start to move it around.
- It turns out that once the test mass has started to move from $L_{4,5}$, the Coriolis force will put it into a stable orbit around $L_{4,5}$. Demonstration in http://www.physics.montana.edu/faculty/cornish/lagrange.pdf



Stability

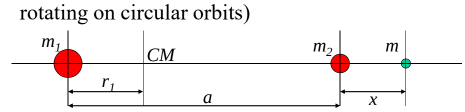
- The Lagrange points L4 and L5 are stable equilibria as long as the heavy body - the Sun in our example - is more than 24.96 times as massive as the intermediate-sized one.
- On the other hand, L1, L2, and L3 are unstable equilibria: the slightest nudge will make satellite at one of these points drift away. To keep it there, you'll need to equip it with thrusters that keep correcting its orbit now and then.
- The stable Lagrange points are the most interesting for astronomy, because stuff tends to accumulate there. For example...

Trojan Asteroids

- So far a total of 1691 asteroids called *Trojans* have been found orbiting the stable Lagrange points of Jupiter's orbit around the sun. In general, the Trojan asteroids at L4 are named after Greek soldiers in the Trojan war, like Achilles, Hektor, Nestor, Agamemnon and Odysseus. Those at L5 are named after guys on the Trojan side, like Patroclus and Priamus.
- There is also one known "Neptune Trojan", named 2001 QR322.
- The asteroid *5261 Eureka* is a "Mars Trojan", occupying the L5 point of Mars' orbit around the sun. It was discovered by David Levy in 1990. Some other Mars Trojans have been discovered since - at least four at L5 and one at L4.
- For the Earth-sun system, there are only *dust clouds* at L4 and L5. They're about four times as big as the Moon. In 1990 the Polish astronomer Winiarski found that they are a few degrees in apparent diameter, that they drift up to ten degrees away from the Lagrange points, and that they're somewhat redder than the usual "zodiacal light" - the light reflecting off dust in the solar system.

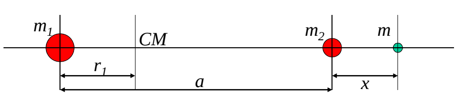
L2

- As an example, let's derive the location of L2, for a test mass m in the system m_1 m_2 (which are rotating on circular orbits)



- If the system rotates at angular speed ω around the center of mass *CM*, the gravitational pulls on *m* and the centrifugal force compensate when

$$\frac{Gmm_1}{(a+x)^2}+\frac{Gmm_2}{x^2}=m\omega^2(x+a-r_1)$$



$$\frac{Gmm_1}{(a+x)^2}+\frac{Gmm_2}{x^2}=m\omega^2(x+a-r_1)$$

- The angular speed is related to the period T:

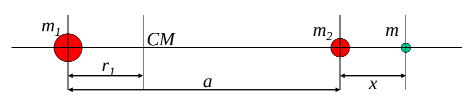
$$\omega^2=\frac{4\pi^2}{T^2}=\frac{G(m_1+m_2)}{a^3}$$

- So we get

$$\frac{Gmm_1}{(a+x)^2}+\frac{Gmm_2}{x^2}=\frac{G(m_1+m_2)}{a^3}m(x+a-r_1)$$

- and using the definition of center of mass

$$m_1r_1=(a-r_1)m_2 \quad\rightarrow\quad r_1=\frac{k}{1+k}a \quad \text{with} \quad k=\frac{m_2}{m_1}$$



- We get an equation for x in terms of *a* and $k=m_2/m_1$:

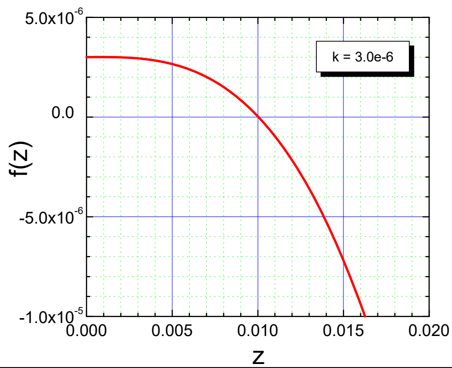
$$m_1x^2a^3+m_2(a+x)^2a^3=(m_1+m_2)(x+a-r_1)(a+x)^2x^2$$

- and defining $z=x/a$:

$$f(z)=z^2+k(1+z)^2-[(z+1)(k+1)-k](1+z)^2z^2=0$$

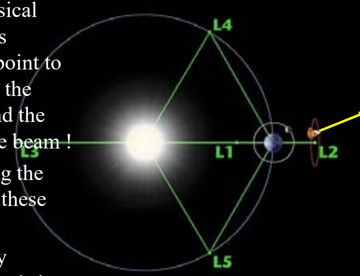
- This fifth-order equation can be solved graphically once the value of *k* is specified.
- For the earth-moon system $k=\dfrac{7.35\times10^{22}}{5.98\times10^{24}}=1.22\times10^{-2}$
- For the sun-earth system $k=\dfrac{5.98\times10^{24}}{1.99\times10^{30}}=3.0\times10^{-6}$

Sun-Earth L2: z=0.01, x=1.5Mkm

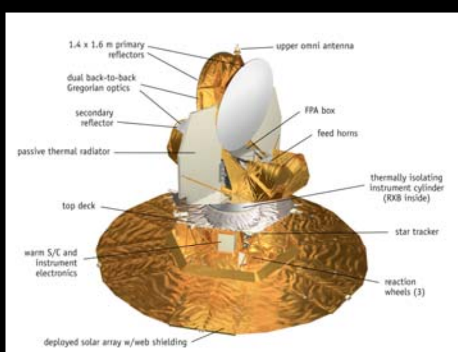


L2

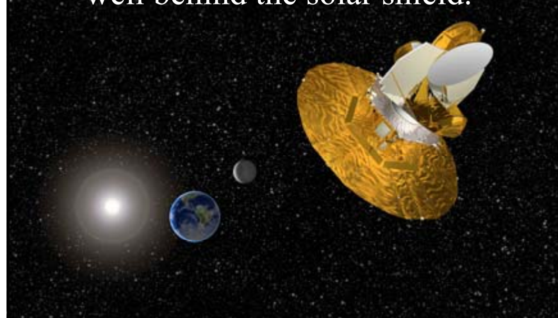
- Earth-Sun L2 is a great place for astrophysical satellites: from this location they can point to deep space having the earth, the moon and the sun all far from the beam !
- Moreover shielding the telescope from all these sources is easy !
- It has been used by WMAP (NASA) and since July 2009 by Herschel and Planck (ESA).

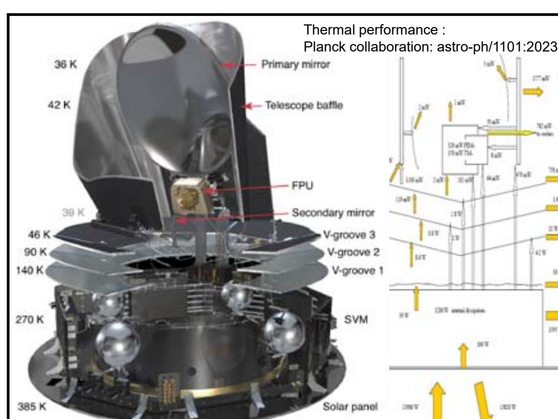


WMAP



WMAP in L_2 : sun, earth, moon are all well behind the solar shield.





L2 and sidelobes

- This is important because WMAP (as the other CMB anisotropy and polarization experiments) measures signals which are extremely faint with respect to the thermal emission of the sun, the moon and the earth.
- For a ground-based experiment, ground pickup is a very important problem.
- But from L2 the earth fills only a very small fraction of the instrument angular response :

$$\theta \approx \frac{d_E}{D_{L2}} = \frac{6360\ km}{1.5\times10^6\ km} \approx 0.24^o$$

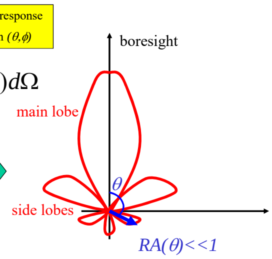
Importance of low sidelobes

- The power detected is the integral of the brightness times the solid angle, weighted with the angular response of the telescope:

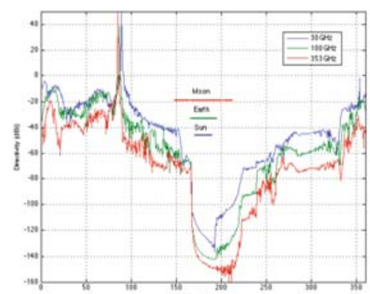
Brightness from direction (θ,ϕ) — Telescope response in direction (θ,ϕ)

$$W = A\int_{4\pi} B(\theta,\varphi)RA(\theta,\varphi)d\Omega$$

- Typical telescope response $RA(\theta,\phi)$

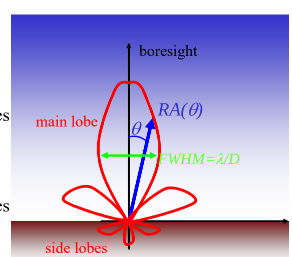


Planck Sidelobes (Tauber et al. A&A 520, A2, 2010, Planck pre-launch status: the optical system)



Importance of low sidelobes

- In the case of CMB observations, the detected brightness is the sum of the brightness from the sky (dominant for the solid angles directed towards the sky, in the main lobe) and the Brightness from ground (dominant for the solid angles directed towards ground, in the sidelobes).



$$W = A\left[\int_{\substack{main\\lobe}} B_{sky}(\theta,\varphi)RA(\theta,\varphi)d\Omega + \int_{\substack{side\\lobes}} B_{Ground}(\theta,\varphi)RA(\theta,\varphi)d\Omega\right]$$

Importance of low sidelobes

$$W = A\left[\int_{\substack{main\\lobe}} B_{sky}(\theta,\varphi)RA(\theta,\varphi)d\Omega + \int_{\substack{side\\lobes}} B_{Ground}(\theta,\varphi)RA(\theta,\varphi)d\Omega\right]$$

$$W \approx A\left[\underbrace{B_{sky}(\theta,\varphi)\left\langle RA_{\substack{main\\lobe}}(\theta,\varphi)\right\rangle\Omega_{\substack{main\\lobe}}}_{\text{signal of interest}} + \underbrace{B_{Ground}(\theta,\varphi)\left\langle RA_{\substack{side\\lobes}}(\theta,\varphi)\right\rangle\Omega_{\substack{side\\lobes}}}_{\text{disturbance signal}}\right]$$

$\approx 3K$ — ≈ 1 — $<<1\ srad$ — $\approx 300K$ — $\approx 2\pi\ srad$

Obtaining : signal of interest >> disturbance signal requires

$$\left\langle RA_{\substack{main\\lobe}}(\theta,\varphi)\right\rangle >>> \left\langle RA_{\substack{side\\lobes}}(\theta,\varphi)\right\rangle$$

signal of interest | disturbance signal

$$W \approx A\left[\underbrace{B_{sky}(\theta,\varphi)}_{\approx 3K}\underbrace{\left\langle RA_{\substack{main\\lobe}}(\theta,\varphi)\right\rangle}_{\approx 1}\underbrace{\Omega_{\substack{main\\lobe}}}_{<<1\ srad}+\underbrace{B_{Ground}(\theta,\varphi)}_{\approx 300K}\left\langle RA_{\substack{side\\lobes}}(\theta,\varphi)\right\rangle\underbrace{\Omega_{\substack{side\\lobes}}}_{\approx 2\pi\ srad}\right]$$

$$\left\langle RA_{\substack{side\\lobes}}(\theta,\varphi)\right\rangle << \left\langle RA_{\substack{main\\lobe}}(\theta,\varphi)\right\rangle\left[\frac{\Omega_{\substack{main\\lobe}}}{\Omega_{\substack{side\\lobes}}}\right]\left[\frac{B_{sky}(\theta,\varphi)}{B_{Ground}(\theta,\varphi)}\right]\approx\frac{\Omega_{\substack{main\\lobe}}(srad)}{600}$$

FWHM	$\Omega_{mainlobe}$	$\langle RA_{sidelobes}\rangle$
10°	2×10^{-2} srad	$<<4\times10^{-5}$
1°	2×10^{-4} srad	$<<4\times10^{-7}$
10'	7×10^{-6} srad	$<<1\times10^{-8}$
1'	7×10^{-8} srad	$<<1\times10^{-10}$

For a ground-based experiment

!!!

signal of interest | disturbance signal

$$W \approx A\left[\underbrace{B_{sky}(\theta,\varphi)}_{\approx 3K}\underbrace{\left\langle RA_{\substack{main\\lobe}}(\theta,\varphi)\right\rangle}_{\approx 1}\underbrace{\Omega_{\substack{main\\lobe}}}_{<<1\ srad}+\underbrace{B_{Ground}(\theta,\varphi)}_{\approx 300K}\left\langle RA_{\substack{side\\lobes}}(\theta,\varphi)\right\rangle\underbrace{\Omega_{\substack{side\\lobes}}}_{\approx 2\pi\ srad}\right]$$

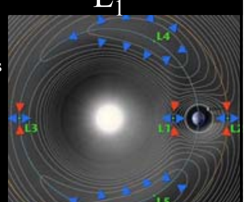
$$\left\langle RA_{\substack{side\\lobes}}(\theta,\varphi)\right\rangle << \left\langle RA_{\substack{main\\lobe}}(\theta,\varphi)\right\rangle\left[\frac{\Omega_{\substack{main\\lobe}}}{\Omega_{\substack{side\\lobes}}}\right]\left[\frac{B_{sky}(\theta,\varphi)}{B_{Ground}(\theta,\varphi)}\right]\approx\frac{\Omega_{\substack{main\\lobe}}(srad)}{2\times10^{-7}\ srad}$$

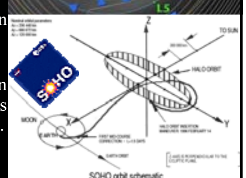
FWHM	$\Omega_{mainlobe}$	$\langle RA_{sidelobes}\rangle$
10°	2×10^{-2} srad	$<<10^{5}$
1°	2×10^{-4} srad	<<1000
10'	7×10^{-6} srad	<<35
1'	7×10^{-8} srad	<<3

For an experiment in L2: the problem is gone, and we can look to signals << 3K, like CMB anisotropy !

- Sun-Earth L_1 can be used for an interrupted view of the sun.
- The ESA-NASA Solar and Heliospheric Observatory (SOHO) was designed to study the internal structure of the Sun, its extensive outer atmosphere and the origin of the solar wind, the stream of highly ionized gas that blows continuously outward through the Solar System.
- It was put into a Halo orbit around L_1 in 1996, where is mantained with the assistance of thrusters.
- SOHO was designed to observe the Sun continuously for at least two years. This is important for heliosismology studies. All previous solar observatories have orbited the Earth, by which the Sun is periodically `eclipsed' .

L_1





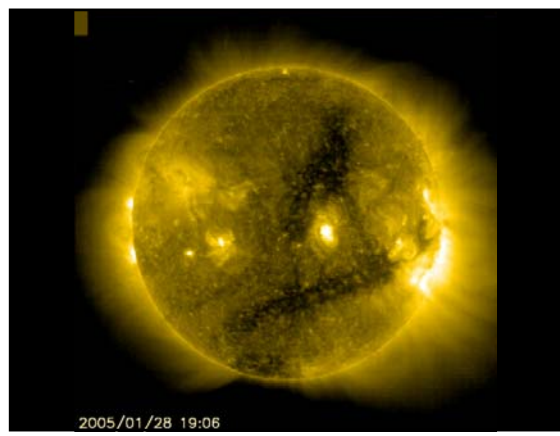


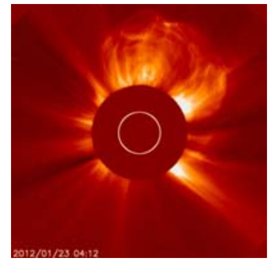
Immagine presa da SOHO (Gennaio 2010)

- Taken on January 3rd,
- an extreme ultraviolet
- Image of the Sun to
- scale, is superimposed
- at the center of the disk.
- Beyond the disk's outer
- boundary is a sungrazer
- comet, one of the
- brightest yet seen
- by SOHO

- http://soho.nascom.nasa.gov/hotshots/index.html/
- Video cometa ISON (27-30 Novembre, 2013)
- ISON made its closest approach to the Sun during the evening of 28 November, passing just 1.2 million kilometres from the Sun's visible surface.

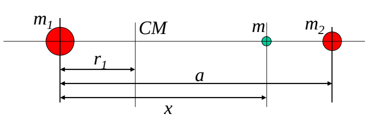
- http://www.nasa.gov/mission_pages/soho/
- While watching the sun's activity, SOHO has produced some discoveries about how the sun works. According to ESA, its chief discoveries include finding complicated gas currents below the sun's visible surface, as well as tracking frequent changes in magnetic fields.
- Researchers came close to losing the spacecraft in June 1998, less than three years after launch, after a routine gyroscope calibration. SOHO was in the wrong position to receive communications from Earth, and was not responding to commands.
- It took about three months to recover, and the spacecraft lost two gyroscopes as a result. Then SOHO's last gyroscope failed, forcing mission managers to develop a new way of keeping SOHO stable.
- SOHO is also a tool in discovering comets, even though it wasn't designed to do so. As of November 2012 the satellite had spotted around 2,400 comets, with new ones (on average) found about once every 2.6 days. Many of those comets were discovered by amateur astronomers downloading images from the SOHO website.

The Solar Heliospheric Observatory (SOHO) captured the coronal mass ejection (CME) from a huge solar flare on Jan. 23, 2012.



L1

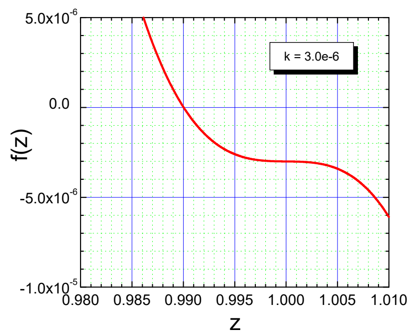
- The location of L1 can be obtained in a way similar to L2 before:



- If the system rotates at angular speed ω around the center of mass *CM*, the gravitational pulls on *m* and the centrifugal force compensate when

$$\frac{Gmm_1}{x^2} - \frac{Gmm_2}{(a-x)^2} = m\omega^2(x - r_1)$$

L1

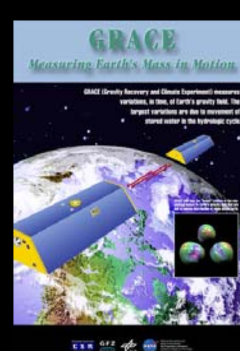


$$f(z) = (1-z)^2 - kz^2 - [z(1+k) - k](1-z)^2 z^2 = 0$$

Earth Oblateness

- Earth is not perfectly spherical
- As a first order approximation it is an oblate spheroid:
  - $R_{equat.}$= 6378.12 km
  - R_{polar}= 6356.74 km
  - Flattening = 1/298.2
- And it is a bit pear-shaped: more mass in the southern hemisphere than in the northern.

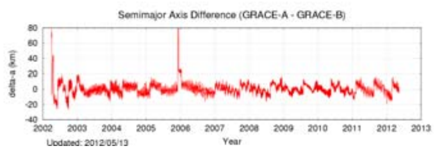
- Satellites are mapping earth gravity and its variations with incredible accuracy: Gravity Recovery And Climate Experiment



http://www.csr.utexas.edu/grace/operations/configuration.html

GRACE satellites were launched on March 17, 2002, on-board Rockot, from Plesetsk Cosmodrome in Siberia. The satellites were injected into a 500 km altitude, near circular polar orbit.

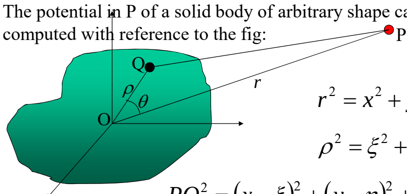
- **Relative Orbit Evolution (plots updated daily)**
- The orbit of GRACE-A relative to GRACE-B is shown in this section. Discounting some early orbit adjustments, the mean semi-major axis difference (shown in the first plot) between the two satellites averages around 0 meters. Step changes in the semi-major axis difference appear when orbit maneuvers are executed in order to keep the separation between 170 and 220 km. In the early days of the mission, some changes were also caused by an attitude mode loss on board one of the spacecrafts, which leads to increased drag.



- https://earth.esa.int/web/guest/missions/3rd-party-missions/current-missions/grace/
- GRACE Operational Issue: Planned switch-on of Microwave Assemblies
- 05 August 2014

Earth Oblateness

- As a result of oblateness, the earth gravitational field is not that of a central point mass.
- This means that for satellites orbiting the earth (and especially for LEO, low earth orbits) there will be a perturbation with respect to the keplerian 2-body solution.
- The potential in P of a solid body of arbitrary shape can be computed with reference to the fig:

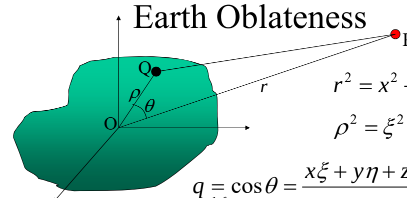


$$r^2 = x^2 + y^2 + z^2$$

$$\rho^2 = \xi^2 + \eta^2 + \zeta^2$$

$$PQ^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$$

Earth Oblateness



$$r^2 = x^2 + y^2 + z^2$$

$$\rho^2 = \xi^2 + \eta^2 + \zeta^2$$

$$q \underset{def}{=} \cos\theta = \frac{x\xi + y\eta + z\zeta}{r\rho} \quad ; \alpha \underset{def}{=} \frac{\rho}{r}$$

$$PQ^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 =$$

$$= r^2\left[1 - 2\left(\frac{x\xi + y\eta + z\zeta}{r\rho}\right)\frac{\rho}{r} + \left(\frac{\rho}{r}\right)^2\right] = r^2\left[1 - 2q\alpha + \alpha^2\right]$$

$$U = G\int_V \frac{dM}{PQ} = G\int_V \frac{dM}{r\sqrt{1 - 2q\alpha + \alpha^2}}$$

Earth Oblateness

$$U = G\int_V \frac{dM}{PQ} = G\int_V \frac{dM}{r\sqrt{1 - 2q\alpha + \alpha^2}}$$

- Since $\alpha < 1 ; q \le 1$
- the square root can be expanded in series to find

$$U = \frac{G}{r}\left[\int_V P_o dM + \int_V P_1 \alpha dM + \int_V P_2 \alpha^2 dM + \ldots + \int_V P_n \alpha^n dM + ..\right]$$

- where the P_i are Legendre polinomials:

$$P_o = 1$$
$$P_1 = q$$
$$P_2 = \tfrac{1}{2}\left(3q^2 - 1\right)$$
$$P_3 = \tfrac{1}{2}\left(5q^3 - 3q\right)$$
...

$$U = U_0 + U_1 + U_2 + \ldots =$$
$$= \frac{GM}{r} + \frac{G}{r}\int_V q\,\alpha\, dM +$$
$$+ \frac{G}{r}\int_V \tfrac{1}{2}\left(3q^2 - 1\right)\alpha^2 dM + \ldots$$

Earth Oblateness

$$U = U_0 + U_1 + U_2 + \ldots =$$
$$= \frac{GM}{r} + \frac{G}{r}\int_V q\,\alpha\, dM + \frac{G}{r}\int_V \tfrac{1}{2}\left(3q^2 - 1\right)\alpha^2 dM + \ldots$$

- U1 is zero because the center of mass is defined so that

$$\int_V \xi dM = 0 \quad ; \int_V \eta dM = 0 \quad ; \int_V \zeta dM = 0$$

- and

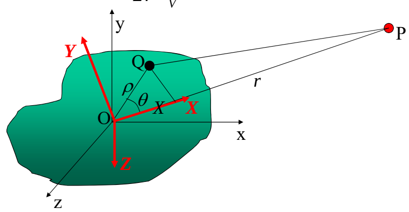
$$U_1 = \frac{G}{r}\int_V q\,\alpha\, dM = \frac{G}{r^2}\int_V \frac{x\xi + y\eta + z\zeta}{r} dM =$$
$$= \frac{Gx}{r^3}\int_V \xi dM + \frac{Gy}{r^3}\int_V \eta dM + \frac{Gz}{r^3}\int_V \zeta dM$$

Earth Oblateness

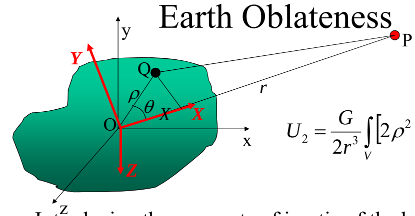
$$U_2 = \frac{G}{r^3}\int_V \tfrac{1}{2}\left(3q^2-1\right)\rho^2 dM = \frac{G}{2r^3}\left[3\int_V X^2 dM - \int_V \rho^2 dM\right]$$

- Where X is the projection of ρ on OP. If Y and Z form a rectangular set with X, then $\rho^2 = X^2+Y^2+Z^2$ and

$$U_2 = \frac{G}{2r^3}\int_V \left[2\rho^2 - 3\left(Y^2+Z^2\right)\right]dM$$



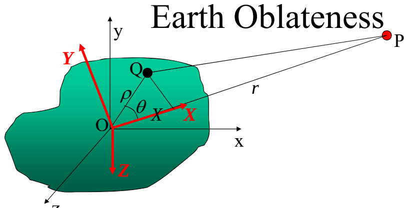
Earth Oblateness



$$U_2 = \frac{G}{2r^3}\int_V \left[2\rho^2 - 3\left(Y^2+Z^2\right)\right]dM$$

- Introducing the moments of inertia of the body around the axes Ox,Oy,Oz and OP:
  $A = \int_V \left(y^2+z^2\right)dM$; $B = \int_V \left(z^2+x^2\right)dM$; $C = \int_V \left(x^2+y^2\right)dM$;
  $I = \int_V \left(Y^2+Z^2\right)dM$
- We can rewrite $U_2 = \frac{G}{2r^3}\left(A+B+C-3I\right)$

Earth Oblateness



- Summarizing: $U \cong \frac{GM}{r} + \frac{G}{2r^3}\left(A+B+C-3I\right)$
- This is called MacCullagh's formula, and is sufficiently accurate for most astronomical purposes.
- The U3 term is zero if the body is symmetrical about all three coordinate planes (for example a homogeneous ellipsoid with three unequal axes)

Earth Oblateness

- Introducing now polar coordinates r, λ (latitude), ϕ (longitude) for dM, we have
  $$\begin{cases} x = r\cos\lambda\cos\phi \\ y = r\sin\lambda\cos\phi \\ z = r\sin\phi \end{cases}$$
- and MacCullagh's formula becomes:
  $$U \cong \frac{GM}{r} + \frac{G}{2r^3}\left(A+B+C-3I\right) =$$
  $$= \frac{GM}{r} + \frac{G}{r^3}\left[\left(C - \frac{A+B}{2}\right)\left(\frac{1}{2} - \frac{3}{2}\sin^2\phi\right) - \frac{3}{4}\left(A-B\right)\cos^2\phi\cos 2\lambda\right]$$
- If the body is rotationally symmetric around the z axis (as the Earth, to a good approximation), then A=B so that
  $$U \cong \frac{GM}{r} + \frac{G}{r^3}\left(C-A\right)\left(\frac{1}{2} - \frac{3}{2}\sin^2\phi\right) \underset{def}{=} \frac{GM}{r}\left[1 - \left(\frac{R}{r}\right)^2 J_2 P_2\left(\sin\phi\right)\right]$$
- For the Earth:
  R = 6360 km, and $J_2 = 1.082\text{x}10^{-3}$.

- This perturbed potential affects the orbit of artificial satellites in different ways.
- It is found that both periodic and secular perturbations arise for the orbit elements. The results for the secular perturbations are:

$$\varpi = \omega_0 + \frac{3}{2}\frac{J_2 R^2 \bar{n}}{p^2}\left(2 - \frac{5}{2}\sin^2 i\right)t$$
Perigee precession

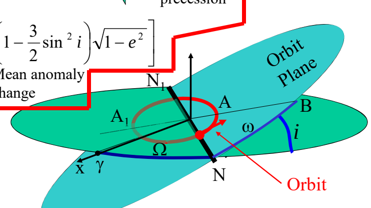
$$\overline{\Omega} = \Omega_0 - \frac{3}{2}\frac{J_2 R^2 \bar{n}}{p^2}\left(\cos i\right)t$$
Orbit plane precession

Secular changes

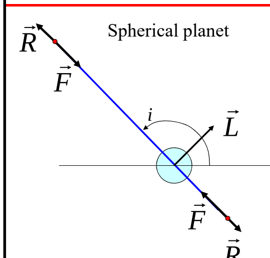
$$\bar{n} = n_0\left[1 + \frac{3}{2}\frac{J_2 R^2 \bar{n}}{p^2}\left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\right]$$

$$M = M_0 + \bar{n}t$$
Mean anomaly change

$$\bar{a} = a_0$$
$$\bar{e} = e_0$$
$$\bar{i} = i_0$$
Only periodic changes

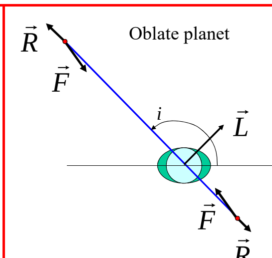


Orbit Precession : $\frac{d\vec{L}}{dt} = \vec{M}$



$$\vec{M} = \vec{F}\times\vec{R} = 0$$

M is 0, so there is no variation of L(m.ang.) which will stay always in the figure plane: the plane of the orbit remains constant.



$$\vec{M} = \vec{F}\times\vec{R} \neq 0$$

M enters the figure plane, so there is a variation of L which will also enter the figure plane: this means that the plane of the orbit precesses.

Orbit Precession

- The size of the effect can be evaluated from

$$\frac{d\overline{\Omega}}{dt} = -\frac{3}{2}\frac{J_2 R^2 \overline{n}}{p^2}\cos i$$

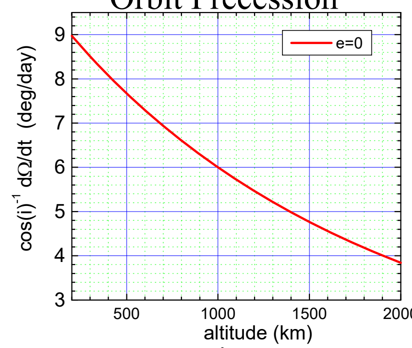
- But $p = a(1-e^2)$; $\overline{n} = \frac{2\pi}{T} = \sqrt{\frac{GM}{a^3}}$
- So that

$$\frac{d\overline{\Omega}}{dt} = -\frac{3}{2} J_2 \frac{1}{(1-e^2)^2}\left(\frac{R}{a}\right)^2 \sqrt{\frac{GM}{a^3}}\cos i$$

- For example for a satellite on a circular orbit at altitude h, we have $a=R+h$, and

$$-\frac{1}{\cos i}\frac{d\overline{\Omega}}{dt} = \frac{3}{2} J_2 \frac{1}{(1-e^2)^2}\left(\frac{R_E}{R_E+h}\right)^2 \sqrt{\frac{GM}{(R_E+h)^3}}$$

Orbit Precession



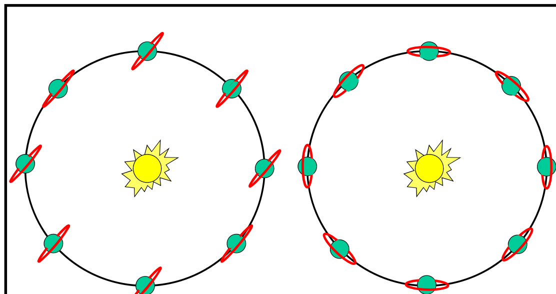
$$-\frac{1}{\cos i}\frac{d\Omega}{dt} = \frac{3}{2} J_2 \frac{1}{(1-e^2)^2}\left(\frac{R_E}{R_E+h}\right)^2 \sqrt{\frac{GM}{(R_E+h)^3}}$$

Sun-Synchronous Orbit

- An orbit whose plane normal points always towards the sun can be very useful (to illuminate optimally the solar panels, or to shield constantly an instrument behind a sun-shield)
- This can easily be obtained by imposing that the orbit precession is 360 deg in one year, i.e. 360/365.25 deg/day=0.9856 deg/day.
- For example, for a satellite in a circular orbit at 900 km of altitude, the condition above writes (see fig.): $\left.\frac{d\Omega}{dt}\right|_{h=900\ km} = 0.9856\ \text{deg/day} \rightarrow \cos i = -\frac{0.9856}{6.293}$

$$\rightarrow \boxed{i = 99.01^o}$$

- This orbit (9 deg in excess of a polar orbit) has been used e.g. for the IRAS satellite



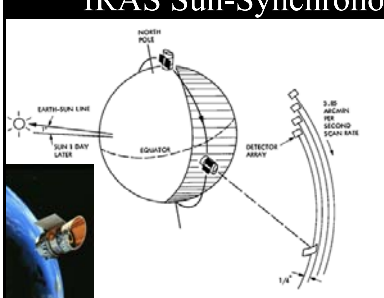
Polar orbit

$\frac{d\Omega}{dt} = 0$

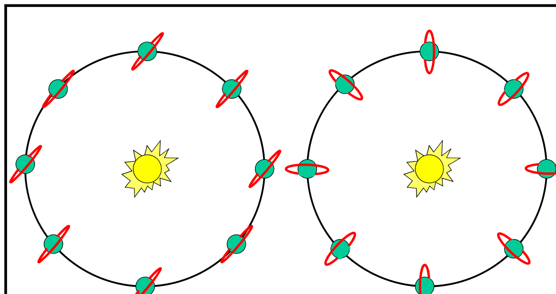
Thermally-stable Sun-synchronous orbit (IRAS)

$\frac{d\Omega}{dt} = 0.9856\ \text{deg/day}$

IRAS Sun-Synchronous Orbit



- All ecliptic latitudes would be swept out during one orbit while, as the line of nodes precessed,all ecliptic longitudes would be covered in a period of six months
- The orbital altitude, 900 km, and inclination, 99°, combined with the Earth's equatorial bulge lead to a precession of the plane of the orbit about 1° per day.
- With the telescope pointing radially outwards from the Earth and perpendicular to the Sun vector, no Earth or sunlight could enter the telescope



Polar orbit

$\frac{d\Omega}{dt} = 0$

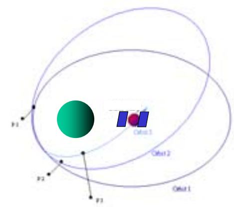
Eclipsing Sun-synchronous orbit

$\frac{d\Omega}{dt} = 0.9856\ \text{deg/day}$

Perigee precession

$$\frac{d\varpi}{dt}=\frac{3}{2}\frac{J_2R^2\bar{n}}{p^2}\left(2-\frac{5}{2}\sin^2 i\right)$$

- The numerical factor in front of the Legendre polinomial is the same as in the Orbit precession. So it is a degs/day effect, depending on the selected inclination.
- There is a critical inclination which makes the perigee precession vanish:
  $$2-\frac{5}{2}\sin^2 i_c=0\rightarrow \boxed{i_c=63.43^o}$$
- It was used in the so-called *Molniya orbit*, used for russian communication satellites. Due to the high latitude of Russia, geostationary satellites are not effective (and they require more fuel to be injected in orbit)



- So they used 12h very elliptic orbits (e=0.7), with $i=i_c$, with apogee above Russia and perigee in the southern hemisphere, so that most of the time (9h or so) the satellites are available for communications in Russia.
- This can only work if the perigee does not precess !